

THE ODE METHOD FOR SOME SELF-INTERACTING DIFFUSIONS ON \mathbb{R}^d

ALINE KURTZMANN

ABSTRACT. The aim of this paper is to study the long-term behaviour of a class of self-interacting diffusion processes on \mathbb{R}^d . These are solutions to SDEs with a drift term depending on the actual position of the process and its normalized occupation measure μ_t . These processes have so far been studied on compact spaces by Benaïm, Ledoux and Raimond, using stochastic approximation methods. We extend these methods to \mathbb{R}^d , assuming a confinement potential satisfying some conditions. These hypotheses on the confinement potential are required since in general the process can be transient, and is thus very difficult to analyze. Finally, we illustrate our study with an example on \mathbb{R}^2 .

1. INTRODUCTION

This paper addresses the long-term behavior of a class of ‘self-interacting diffusion’ processes $(X_t, t \geq 0)$ on non-compact spaces. These processes are time-continuous, non-Markov and live on \mathbb{R}^d . They are solutions to a kind of diffusion SDEs, whose drift term depends on the whole past of the path through the occupation measure of the process. Due to their non-Markovianity, they often exhibit an interesting ergodic behavior.

1.1. Previous results on self-interacting diffusions. Time-continuous self-interacting processes, also named ‘reinforced processes’, have already been studied in many contexts. Under the name of ‘Brownian polymers’, Durrett & Rogers [9] first introduced them as a possible mathematical model for the evolution of a growing polymer. They are solutions of SDEs of the form

$$dX_t = dB_t + dt \int_0^t ds f(X_t - X_s)$$

where $(B_t; t \geq 0)$ is a standard Brownian motion on \mathbb{R}^d and f is a given function. As the process $(X_t; t \geq 0)$ evolves in an environment changing with its past trajectory, this SDE defines a self-interacting diffusion, either self-repelling or self-attracting, depending on f .

Another modelisation, with dependence on the (convoled) normalized occupation measure $(\mu_t, t \geq 0)$, has also been considered since the work of Benaïm, Ledoux & Raimond [4]. They introduced a process living in a compact smooth connected Riemannian manifold M without boundary:

$$(1.1) \quad dX_t = \sum_{i=1}^N F_i(X_t) \circ dB_t^i - \int_M \nabla_x W(X_t, y) \mu_t(dy) dt,$$

where W is a (smooth) interaction potential, (B^1, \dots, B^N) is a standard Brownian motion on \mathbb{R}^N and the symbol \circ stands for the Stratonovich stochastic integration. The family of smooth vector fields $(F_i)_{1 \leq i \leq N}$ comes from the Hörmander ‘sum of squares’ decomposition of the Laplace-Beltrami operator $\Delta = \sum_{i=1}^N F_i^2$. The normalized occupation measure μ_t is defined by:

$$(1.2) \quad \mu_t := \frac{r}{r+t} \mu + \frac{1}{r+t} \int_0^t \delta_{X_s} ds$$

This work is partially supported by the Swiss National Foundation grants 200020-112316/1 and PBNE2-119027. The author is glad to thank M. Benaïm for introducing her to the subject, P. Cattiaux for an introduction to the ultracontractivity property, V. Kleptsyn and P. Tarrès for helpful comments, and J.-F. Jouanin for valuable discussions.

where μ is an initial probability measure and r is a positive weight. In the compact-space case, they showed that the asymptotic behavior of μ_t can be related to the analysis of some deterministic dynamical flow defined on the space of the Borel probability measures. They went further in this study in [5] and gave sufficient conditions for the a.s. convergence of the normalized occupation measure. When the interaction is symmetric, μ_t converges a.s. to a local minimum of a nonlinear free energy functional (each local minimum having a positive probability to be chosen). All these results are summarized in a recent survey of Pemantle [17].

The present paper follows the same lead and extends the results of Benaïm, Ledoux & Raimond [4] in the non-compact setting. We present all results in the Euclidean space \mathbb{R}^d for the sake of simplicity, but, they can be extended to the case of a complete connected Riemannian manifold M without boundary with no further difficulty than the use of notations and a bit of geometry. The point is to involve the Ricci curvature in the conditions and work on the space $M \setminus \text{cut}(o)$, where $\text{cut}(o)$ is the cut locus of o (which has zero-mean).

1.2. Statement of the problem. Here we set the main definitions: let us consider a confinement potential $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and an interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$. For any bounded Borel measure μ , we consider the ‘convolved’ function

$$(1.3) \quad W * \mu : \mathbb{R}^d \rightarrow \mathbb{R}, \quad W * \mu(x) := \int_{\mathbb{R}^d} W(x, y) \mu(dy).$$

Our main object of interest is the self-interacting diffusion solution to

$$(1.4) \quad \begin{cases} dX_t = dB_t - (\nabla V(X_t) + \nabla W * \mu_t(X_t)) dt \\ d\mu_t = (\delta_{X_t} - \mu_t) \frac{dt}{r+t} \\ X_0 = x, \mu_0 = \mu \end{cases}$$

where (B_t) is a d -dimensional Brownian motion. Our goal is to study the long-term behavior of $(\mu_t, t \geq 0)$. Let us recall that the main difference with the previous work [4] is that the state space is \mathbb{R}^d and hence is not compact anymore. However, we will be able to extend the results obtain in the compact case: the behavior of μ_t is closely related to the behavior of a deterministic flow. We will also give some sufficient conditions on the interaction potential in order to prove ergodic results for the process $(X_t, t \geq 0)$.

Before stating the theorems proved in this paper, let us briefly describe the main results obtained so far on self-interacting diffusions in non-compact spaces. They concern the model of Durrett & Rogers, and can be classified in three categories. The first one is when f is real, compactly supported and its sign is constant. Cranston & Mountford [7] have solved a (partially proved) conjecture of Durrett & Rogers and shown that X_t/t converges a.s. The second one deals with attracting interaction on \mathbb{R} (i.e. $xf(x) \leq 0$ for all $x \in \mathbb{R}$) studied in the constant case by Cranston & Le Jan [6] and its generalization by Raimond [18] in the d -dimensional “constant” case $f(x) = -ax/\|x\|$, or by Herrmann & Roynette [10] for a local interaction. Under some conditions, it is proved that X_t converges a.s., whereas for a non-local interaction, it does not in general (but the paths are a.s. bounded for $f(x) = -\text{sign}(x)\mathbb{1}_{|x| \geq a}$). The third one concerns a non-integrable repulsive f on \mathbb{R} (i.e. $xf(x) \geq 0$ for all $x \in \mathbb{R}$) studied by Mountford & Tarrès [16] and solving a conjecture of Durrett & Rogers. They have proved that for $f(x) = x/(1 + |x|^{1+\beta})$, with $0 < \beta < 1$, there exists a positive c such that with probability 1/2, the symmetric process $t^{-2/(1+\beta)}X_t$ converges to c .

These previous works have in common that the drift may overcome the noise, so that the randomness of the process is “controlled”. To illustrate that, let us mention, for the same model of Durrett & Rogers, the case of a repulsive function f of compact support, also conjectured in [9], which is still unsolved.

Conjecture 1. [9] *Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an odd function, of compact support. Then X_t/t converges a.s. to 0.*

Coming back to our process of interest, the role of the confinement potential is to similarly “control” the drift term of the diffusion. Indeed, for the process (1.4) with $V = 0$, the interaction potential is in general not strong enough for the process to be recurrent, and the behavior is then very difficult to analyze. In particular, it is hard to predict the relative importance of the drift term (in competition against the Brownian motion) in the evolution.

1.2.1. *Technical assumptions on the potentials.* In the sequel, (\cdot, \cdot) stands for the Euclidian scalar product. We denote by **(H)** the following hypotheses:

- i) (*regularity and positivity*) $V \in \mathcal{C}^2(\mathbb{R}^d)$ and $W \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $V \geq 1$, $W \geq 0$;
- ii) (*convexity*) V is a strictly uniformly convex function: there exists $K > 0$ such that for all $x, \xi \in \mathbb{R}^d$: $(\nabla^2 V(x)\xi, \xi) \geq K|\xi|^2$;
- iii) (*growth*) there exist $c, C > 0$, $\delta > 1$ such that for all x large enough, $(\nabla V(x), x) \geq c|x|^{2\delta}$ and for all $x, y \in \mathbb{R}^d$

$$(1.5) \quad |\nabla V(x) - \nabla V(y)| \leq C(|x - y| \wedge 1)(V(x) + V(y));$$

- iv) (*domination*) there exists $\kappa \geq 1$ such that for all $x, y \in \mathbb{R}^d$,

$$(1.6) \quad W(x, y) + |\nabla_x W(x, y)| + |\nabla_{xx}^2 W(x, y)| \leq \kappa(V(x) + V(y));$$

- v) (*curvature*) there exist $\alpha > -1$, $M \in \mathbb{R}$ such that for all $x, y, \xi \in \mathbb{R}^d$

$$(1.7) \quad \frac{(x, \nabla_x W(x, y))}{(x, \nabla V(x))} \rightarrow \alpha \text{ and } ((\nabla^2 V(x) + \nabla_{xx}^2 W(x, y))\xi, \xi) \geq M|\xi|^2.$$

Remark 1.1. 1) The most important conditions are the domination iv) and the curvature v).
2) The growth condition (1.5) on V ensures that there exists $a > 0$ such that for all $x \in \mathbb{R}^d$, we have

$$(1.8) \quad \Delta V(x) \leq aV(x).$$

3) The positivity and domination conditions (1.6) on the interaction potential are not so hard to be satisfied, since the self-interacting process will be invariant by the gauge transform $W(x, y) \mapsto W(x, y) + \phi(y)$ for any function ϕ that does not grow faster than V .

1.2.2. *Results.* We can now describe the behavior of μ_t .

Theorem 1.2. Suppose **(H)**.

- 1) $\mathbb{P}_{x,r,\mu}$ -a.s., the ω -limit set of $(\mu_t, t \geq 0)$ is weakly compact, invariant by Φ and admits no other (sub-)attractor than itself.
- 2) If W is symmetric, then, $\mathbb{P}_{x,r,\mu}$ -a.s., the ω -limit set of $(\mu_t, t \geq 0)$ is a connected subset of set of fixed points of the probability measure proportional to $e^{-(V+W*\mu)(x)} dx$.

Even if the model studied could at a first glance seem restrictive (because of V), the drift term can really compete against the Brownian motion, as shown by the following:

Theorem 1.3. Consider the self-interacting diffusion on \mathbb{R}^2 , with $V(x) = V(|x|)$ and $W(x, y) = (x, Ry)$. Let $\gamma(\rho) := e^{-2V(\rho)}/Z$. Then one of the following holds:

- (1) If V is such that $\int_0^\infty d\rho \gamma(\rho) \rho^2 \cos(\theta) > -1$, then a.s. $\mu_t \xrightarrow{(w)} \gamma$;
- (2) Else, we get two different cases:

a) if $\theta = \pi$ then there exists a random measure μ_∞ such that a.s. $\mu_t \xrightarrow{(w)} \mu_\infty$,

b) if $\theta \neq \pi$, then the ω -limit set $\omega(\mu_t, t \geq 0) = \{\nu(\delta), 0 \leq \delta < 2\pi\}$ a.s., that is μ_t circles around.

1.3. Outline of contents. As mentioned earlier, the main difficulty here stems from the non-compactness of the state space. The way to get around it is first to introduce the V -norm in Section 2 (also named ‘dual weighted norm’) and then show that $(\mu_t, t \geq 0)$ is a tight family of measures in §6.1. Second, the dynamical system involved in the study induces only a local semiflow and not necessarily a global one. But, we will show in §4.2 (for some cases), that the semiflow does not explode. Last, define the Feller diffusion X^μ obtained by fixing the occupation measure μ_t (appearing in the drift term) to μ . Let note A_μ its infinitesimal generator and Q_μ its fundamental kernel, that is $A_\mu \circ Q_\mu = \Pi(\mu) - Id$, where $\Pi(\mu)$ is the invariant probability measure of X^μ . An essential point of our study consists in finding an upper bound for the operator Q_μ . Indeed, one has to use here the notion of (uniform) ultracontractivity in §5.1.

The organization of this paper is as follows. In the next section, we introduce some notations and the deterministic flow involved. We will also prove the existence and uniqueness of the random process studied. Section 3 is devoted to the presentation of the main results and is divided in three parts. First, we recall the former results and ideas of Benaïm and *al* [4]. Then, we state the tightness of $(\mu_t)_t$ and introduce the uniform estimates on the Feller semigroup. We end this section by describing the behavior of μ_t . After that, we analyze, in Section 4, the deterministic semiflow associated to the self-interacting diffusion. We will prove the local existence of the semiflow and introduce two important objects: the convex hull of $\text{Im}(\Pi)$ and the fixed points of Π . Then, in Section 5, we study in details the family of Markov semigroups, corresponding to X^μ , for which we prove the uniform ultracontractivity property and the regularity of the operators A_μ and Q_μ . The proofs of the main results are in Section 6, which heavily relies on the spectral analysis of the preceding section. We begin Section 6 by showing the tightness of $(\mu_t)_t$ in §6.1. Then, §6.2 deals with the approximation of the normalized occupation measure $(\mu_t, t \geq 0)$ by a deterministic semiflow. In §6.3, we prove Theorem 1.2. Finally, Section 7 is devoted to the illustration in dimension $d = 2$ stated in Theorem 1.3.

2. PRELIMINARIES AND TOOLS

2.1. Some useful measure spaces. We denote by $\mathcal{M}(\mathbb{R}^d)$ the space of signed (bounded) Borel measures on \mathbb{R}^d and by $\mathcal{P}(\mathbb{R}^d)$ its subspace of probability measures. We will need the following measure space:

$$(2.1) \quad \mathcal{M}(\mathbb{R}^d; V) := \{\mu \in \mathcal{M}(\mathbb{R}^d); \int_{\mathbb{R}^d} V(y)|\mu|(dy) < \infty\},$$

where $|\mu|$ is the variation of μ (that is $|\mu| := \mu^+ + \mu^-$ with (μ^+, μ^-) the Hahn-Jordan decomposition of μ). This space will enable us to always check the integrability of V (and therefore of W and its derivatives thanks to the domination condition (1.6)) with respect to the measures to be considered. For example it contains the measure

$$(2.2) \quad \gamma(dx) := \exp(-2V(x))dx.$$

We endow $\mathcal{M}(\mathbb{R}^d; V)$ with the following dual weighted supremum norm (or dual V -norm) defined by

$$(2.3) \quad \|\mu\|_V := \sup_{\varphi; |\varphi| \leq V} \left| \int_{\mathbb{R}^d} \varphi d\mu \right|, \quad \mu \in \mathcal{M}(\mathbb{R}^d; V).$$

This norm naturally arises in the approach to ergodic results for time-continuous Markov processes of Meyn & Tweedie [15]. It makes $\mathcal{M}(\mathbb{R}^d; V)$ a Banach space. To illustrate the need of this space, we state an easy result that will be used many times:

Lemma 2.1. *For any $\mu \in \mathcal{M}(\mathbb{R}^d; V)$ the function $W * \mu$ belongs to $\mathcal{C}^2(\mathbb{R}^d)$ and we have the estimate*

$$|W * \mu(x)| \leq 2\kappa \|\mu\|_V V(x).$$

Proof. Straightforward thanks to the domination condition (1.6). □

In the following, we will consider probability measures only. Thus we set, $\mathcal{P}(\mathbb{R}^d; V) := \mathcal{M}(\mathbb{R}^d; V) \cap \mathcal{P}(\mathbb{R}^d)$. The strong topology on $\mathcal{P}(\mathbb{R}^d; V)$ is the trace topology of the one defined on $\mathcal{M}(\mathbb{R}^d; V)$. It makes $\mathcal{P}(\mathbb{R}^d; V)$ a complete metric space (for the norm distance). Finally, for any $\beta > 1$, we introduce the subspace

$$(2.4) \quad \mathcal{P}_\beta(\mathbb{R}^d; V) := \{\mu \in \mathcal{P}(\mathbb{R}^d); \int_{\mathbb{R}^d} V(y)\mu(dy) \leq \beta\}.$$

2.2. The family of semigroups (P_t^μ) . In all the following, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ will be a filtered probability space satisfying the usual conditions. For any Borel probability measure $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, let $(X_t^\mu, t \geq 0)$ be the Feller diffusion defined by the following SDE

$$(2.5) \quad \begin{cases} dX_t^\mu = dB_t - (\nabla V(X_t^\mu) + \nabla W * \mu(X_t^\mu)) dt, \\ X_0^\mu = x. \end{cases}$$

Proposition 2.2. *The diffusion X_t^μ a.s. never explodes.*

Proof. It is enough to check with the Itô formula that the function

$$(2.6) \quad \mathcal{E}_\mu(x) := V(x) + W * \mu(x).$$

is a Lyapunov function. To see it, we notice that the growth and domination conditions (1.5) and (1.6) on the potentials imply the existence of $D > 0$ such that:

$$(2.7) \quad \Delta \mathcal{E}_\mu(x) \leq 2DV(x) \leq 2D\mathcal{E}_\mu(x).$$

As a by-product we get the naive (but useful!) estimate

$$(2.8) \quad \mathbb{E}\mathcal{E}_\mu(X_t^\mu) \leq \mathcal{E}_\mu(x)e^{Dt}.$$

□

Now we denote by $(P_t^\mu; t \geq 0)$ the Markov semigroup associated to X_t^μ . We consider the differential operator A_μ defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by

$$(2.9) \quad A_\mu f := \frac{1}{2}\Delta f - (\nabla V + \nabla W * \mu, \nabla f)$$

A_μ corresponds to the infinitesimal generator of the true diffusion (2.5). We emphasize that (X_t^μ) is a positive-recurrent (reversible) diffusion and denote by $\Pi(\mu) \in \mathcal{P}(\mathbb{R}^d; V)$ its unique invariant probability measure:

$$(2.10) \quad \Pi(\mu)(dx) := \frac{e^{-2W * \mu(x)}}{Z(\mu)} \gamma(dx)$$

where $Z(\mu) := \int_{\mathbb{R}^d} e^{-2W * \mu(x)} \gamma(dx) < +\infty$ is the normalization constant. To end this part, we recall the classical ergodic theorem for X_t^μ . We introduce the weighted supremum norm (or V -norm)

$$(2.11) \quad \|f\|_V := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{V(x)},$$

and the space of continuous V -bounded functions

$$(2.12) \quad \mathcal{C}^0(\mathbb{R}^d; V) := \{f \in \mathcal{C}^0(\mathbb{R}^d) : \|f\|_V < \infty\}$$

Similarly let $\mathcal{C}^p(\mathbb{R}^d; V) := \mathcal{C}^p(\mathbb{R}^d) \cap \mathcal{C}^0(\mathbb{R}^d; V)$ for all integer $p \geq 1$. The ergodic theorem says that a.s. we have, for all $f \in \mathcal{C}^0(\mathbb{R}^d; V)$:

$$(2.13) \quad \lim_{t \rightarrow \infty} \frac{1}{1+t} \int_0^t f(X_s^\mu) ds = \Pi(\mu)f.$$

2.3. The infinite-dimensional ODE. We introduce a dynamical system on the set of probability measures $\mathcal{P}(\mathbb{R}^d; V)$. We will assume the existence of the semiflow $\Phi : \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d; V) \rightarrow \mathcal{P}(\mathbb{R}^d; V)$ defined by

$$(2.14) \quad \begin{cases} \Phi_t(\mu) = e^{-t}\mu + e^{-t} \int_0^t e^s \Pi(\Phi_s(\mu)) ds, \\ \Phi_0(\mu) = \mu. \end{cases}$$

Remark 2.3. In Section 4, we will prove the local existence of the semiflow, and, for W symmetric or bounded, we will prove it never explodes.

In order to study the semiflow Φ , we will need to endow the space $\mathcal{P}(\mathbb{R}^d; V)$ with different topologies. When nothing else is stated, we will consider that it is endowed with the strong topology defined by the dual weighted supremum norm $\|\cdot\|_V$. But, as the reader will notice, we will frequently need to switch from the strong topology to the weak topology of convergence of measures. We adopt here a non-standard definition compatible with possibly unbounded functions (yet dominated by V): for any sequence of probability measures $(\mu_n, n \geq 1)$ and any probability measure μ (all belonging to $\mathcal{P}(\mathbb{R}^d; V)$), we define the weak convergence as:

$$(2.15) \quad \mu_n \xrightarrow{w} \mu \text{ if and only if } \int_{\mathbb{R}^d} \varphi d\mu_n \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi d\mu, \forall \varphi \in \mathcal{C}^0(\mathbb{R}^d; V).$$

We point out that our definition of the weak convergence is stronger than the usual one. We recall that $\mathcal{P}(\mathbb{R}^d; V)$, equipped with the weak topology, is a metrizable space. Since $\mathcal{C}^0(\mathbb{R}^d; V)$ is separable, we exhibit a sequence $(f_k)_k$ dense in $\{f \in \mathcal{C}^0(\mathbb{R}^d; V) / \|f\|_V \leq 1\}$, and set for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^d; V)$:

$$(2.16) \quad d(\mu, \nu) := \sum_{k=1}^{\infty} 2^{-k} |\mu(f_k) - \nu(f_k)|.$$

Then the weak topology is the metric topology generated by d .

2.4. The self-interacting diffusion. We recall the self-interacting diffusion considered here:

$$\begin{cases} dX_t = dB_t - (\nabla V(X_t) + \nabla W * \mu_t(X_t)) dt \\ d\mu_t = (\delta_{X_t} - \mu_t) \frac{dt}{r+t} \\ X_0 = x, \mu_0 = \mu \end{cases}$$

Proposition 2.4. For any $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ and $r > 0$, there exists a unique global strong solution $(X_t, \mu_t, t \geq 0)$.

Proof. First, we point out that, for all $t > 0$ such that $(X_s, s \leq t)$ is defined, $\mu_t \in \mathcal{P}(\mathbb{R}^d; V)$. In order to show that the solution never explodes, we use again the Lyapunov functional $(x, \mu) \mapsto \mathcal{E}_\mu(x)$ (see (2.6)). As the process $(t, x) \mapsto \mathcal{E}_{\mu_t}(x)$ is of class \mathcal{C}^2 (in the space variable) and is a \mathcal{C}^1 -semi-martingale (in the time variable), the generalized Itô formula (or Itô-Ventzell formula, see [12]), applied to $(t, x) \mapsto \mathcal{E}_{\mu_t}(x)$ implies

$$\begin{aligned} \mathcal{E}_{\mu_t}(X_t) &= \mathcal{E}_\mu(x) + \int_0^t (\nabla \mathcal{E}_{\mu_s}(X_s), dB_s) - \int_0^t |\nabla \mathcal{E}_{\mu_s}(X_s)|^2 ds \\ &\quad + \frac{1}{2} \int_0^t \Delta \mathcal{E}_{\mu_s}(X_s) ds + \int_0^t (W(X_s, X_s) - W * \mu_s(X_s)) \frac{ds}{r+s}. \end{aligned}$$

Let us introduce the sequence of stopping times

$$\tau_n := \inf\{t \geq 0; \mathcal{E}_{\mu_t}(X_t) + \int_0^t |\nabla \mathcal{E}_{\mu_s}(X_s)|^2 ds > n\}.$$

We note that $\int_0^{t \wedge \tau_n} (\nabla \mathcal{E}_{\mu_s}(X_s), dB_s)$ is a true martingale. Again equation (2.6) implies

$$\mathbb{E} \mathcal{E}_{\mu_{t \wedge \tau_n}}(X_{t \wedge \tau_n}) \leq \mathcal{E}_\mu(x) + D \int_0^t \mathbb{E} \mathcal{E}_{\mu_{s \wedge \tau_n}}(X_{s \wedge \tau_n}) ds.$$

Therefore Gronwall's lemma leads to the same kind of estimate as for X^μ :

$$\mathbb{E}\mathcal{E}_{\mu_{t \wedge \tau_n}}(X_{t \wedge \tau_n}) \leq \mathcal{E}_\mu(x)e^{Dt}.$$

As $\lim_{|x| \rightarrow \infty} V(x) = \infty$, the process $(X_t, t \geq 0)$ does not explode in a finite time and the SDE (1.4) admits a global strong solution. \square

3. MAIN RESULTS

3.1. Former tools and general idea. We recall how Benaïm, Ledoux & Raimond handled in [4] the asymptotic behavior of μ_t in a compact space. Indeed, we sketch here the general idea of the study and explain why the tools introduced in the preliminary Section arise quite naturally.

Begin to consider that the occupation measure appearing in the drift is ‘frozen’ to some fixed measure μ . We obtain the Feller diffusion X_t^μ . For this diffusion, it is easy to prove the existence of a spectral gap, and that the semigroup $(P_t^\mu; t \geq 0)$ is exponentially V -uniformly ergodic:

$$(3.1) \quad \|P_t^\mu f - \Pi(\mu)f\|_V \leq K(\mu)\|f\|_V e^{-c(\mu)t}, \quad f \in \mathcal{C}^0(\mathbb{R}^d; V).$$

To get, as by-product, the almost sure convergence of the empirical occupation measure of the process X_t^μ (see (2.13)), a standard technique is to consider the operator (sometimes called the ‘fundamental kernel’ as in Kontoyiannis & Meyn [11]) for any $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$

$$(3.2) \quad Q_\mu f := \int_0^\infty (P_t^\mu f - \Pi(\mu)f) dt$$

Then it is enough to apply the Itô formula to $Q_\mu f(X_t^\mu)$ and divide both members by t to get the desired result. Indeed one has

$$Q_\mu f(X_t^\mu) = Q_\mu f(x) + \int_0^t (\nabla Q_\mu f(X_s^\mu), dB_s) + \int_0^t A_\mu Q_\mu f(X_s^\mu) ds.$$

Some easy bounds on the semigroup (P_t^μ) are enough to prove that almost all terms are negligible compared to t and it remains to recognize the third term since $A_\mu Q_\mu f = \Pi(\mu)f - f$.

Now when μ_t changes in time, we still can write a convenient extended form of the Itô formula (which let appear the time derivative of $Q_{\mu_t}f(x)$) but we need to improve the remainder of the argument. Intuitively, the distance between the time-derivative of μ_{e^t} and the term $\Pi(\mu_{e^t}) - \mu_{e^t}$ converges to zero a.s. As for stochastic approximation processes, one expects the trajectories of μ_t to approximate the trajectories of a deterministic semiflow, meaning that the empirical measure μ_t is an asymptotic pseudotrajectory for the semiflow Φ induced by $\Pi(\mu) - \mu$. This very last remark conveyed to Benaïm & al [4] the idea of comparing the asymptotic evolution of $(\mu_t; t \geq 0)$ with the semiflow $(\Phi_t(\mu))$.

The notion of asymptotic pseudotrajectory was first introduced in Benaïm & Hirsch [3]. It is particularly useful to analyze the long-term behavior of stochastic processes, considered as approximations of solutions of ordinary differential equation (the “ODE method”). Let us give here some definitions.

Definition 3.1. *i) For every continuous function $\xi : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R}^d; V)$, the ω -limit set of ξ , denoted by $\omega(\xi_t, t \geq 0)$, is the set of limits of weak convergent sequences $\xi(t_k), t_k \uparrow \infty$, that is*

$$(3.3) \quad \omega(\xi_t, t \geq 0) := \bigcap_{t \geq 0} \overline{\xi([t, \infty))},$$

where $\overline{\xi([t, \infty))}$ stands for the closure of $\xi([t, \infty))$ according to the weak topology.

ii) A continuous function $\xi : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R}^d; V)$ is an asymptotic pseudotrajectory (or asymptotic pseudo-orbit) for the semiflow Φ if for all $T > 0$,

$$(3.4) \quad \lim_{t \rightarrow +\infty} \sup_{0 \leq s \leq T} d(\xi_{t+s}, \Phi_s(\xi_t)) = 0.$$

The purpose here is to find an asymptotic pseudotrajectory for the semiflow Φ defined by (2.14).

3.2. New tools: tightness and uniform estimates. We will prove that we can find $\beta > 1$ such that $\mu_t \in \mathcal{P}_\beta(\mathbb{R}^d; V)$ $\mathbb{P}_{x,r,\mu}$ -a.s. for t large enough. Remark, that this last set is compact for the weak topology, and so $(\mu_t, t \geq 0)$ is a.s. tight. Then, we have to obtain precise bounds on the family of semigroups $(P_t^\mu, t \geq 0)$ where $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$. Obviously, while using the ergodic estimates (3.1), we would like to get bounds that are uniform in μ .

3.2.1. Tightness. The paper of Benaïm & al [4] crucially relies on the compactness of the manifold M where the diffusion lives. This compactness readily implies that the process $(\mu_t, t \geq 0)$ will very fast be close to the ‘invariant’ probability measure $\Pi(\mu_t)$. On the contrary, if the state space is \mathbb{R}^d and $V \equiv 0$, then X will escape from any compact set. Indeed, the confinement potential V forces the process $(\mu_t, t \geq 0)$ to remain in a (weakly) compact space of measures, for t large, and X is then recurrent. We first exhibit a useful weakly compact set, for $\beta > 0$:

Proposition 3.2. $\mathcal{P}_\beta(\mathbb{R}^d; V)$ is a weakly compact subset of $\mathcal{P}(\mathbb{R}^d; V)$.

Proof. It is clear that $\mathcal{P}_\beta(\mathbb{R}^d; V)$ is weakly closed. The Prohorov theorem shows that it is enough to prove that $\mathcal{P}_\beta(\mathbb{R}^d; V)$ is tight. For every $a > 0$ and $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, we have

$$\min_{|x|>a} V(x) \mu(\{|x| > a\}) \leq \int_{|x|>a} V(x) \mu(dx) \leq \beta.$$

Since $V(x) \rightarrow \infty$ when $|x| \rightarrow \infty$, we see that, for every $\varepsilon > 0$, we can choose a large enough such that $\mu(\{|x| > a\}) \leq \varepsilon$ uniformly in $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$. \square

Proposition 3.3. There exists $\beta > 1$ such that $\mu_t \in \mathcal{P}_\beta(\mathbb{R}^d; V)$ for all t large enough, $\mathbb{P}_{x,r,\mu}$ -a.s.

The proof is postponed to Section 6. Combined with Proposition 3.2, it implies that the family $(\mu_t, t \geq 0)$ is a.s. tight.

3.2.2. Uniform estimates on the semigroup. A priori, it is not obvious (in a non-compact space), that the semigroup (P_t^μ) admits a (uniform) spectral gap. But this is true here. We will indeed prove a stronger result: (P_t^μ) is uniformly ultracontractive, i.e. it is uniformly bounded as an operator from $L^2(\Pi(\mu))$ to L^∞ . Section 5 will be devoted to those uniform properties of the family of semigroups $(P_t^\mu; t, \mu)$.

Proposition 3.4. The family of semigroups $(P_t^\mu, t \geq 0, \mu \in \mathcal{P}_\beta(\mathbb{R}^d; V))$ is uniformly ultracontractive: there exists $c > 0$ independent from μ such that for all $1 \geq t > 0$ and $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, we have

$$(3.5) \quad \|P_t^\mu\|_{2 \rightarrow \infty} := \sup_{f \in C^\infty(\mathbb{R}^d; V) \setminus \{0\}} \frac{\|P_t^\mu f\|_\infty}{\|f\|_{2,\mu}} \leq \exp\left(ct^{-\delta/(\delta-1)}\right).$$

Corollary 3.5. The family of measures $(e^{-2W^*\mu(x)} \gamma(dx), \mu \in \mathcal{P}_\beta(\mathbb{R}^d; V))$, satisfies a logarithmic Sobolev inequality and there exists a uniform spectral gap for the family of measures $(\Pi(\mu), \mu \in \mathcal{P}_\beta(\mathbb{R}^d; V))$. It corresponds to: $\exists C, C_1, C_2 > 0$, independent of μ , such that $\forall f \in C^\infty(\mathbb{R}^d; V)$:

$$\begin{aligned} i) \quad & \int f^2 \log\left(\frac{f^2}{\|f\|_{2,\mu}^2}\right) e^{-2W^*\mu} d\gamma \leq C_2 \int |\nabla f|^2 e^{-2W^*\mu} d\gamma. \\ ii) \quad & \int f^2 d\Pi(\mu) - \left(\int f d\Pi(\mu)\right)^2 \leq C_1 \int |\nabla f|^2 d\Pi(\mu). \end{aligned}$$

Furthermore, $\forall t \geq 0$, $\|P_t^\mu(K_\mu f)\|_{2,\mu} \leq e^{-t/C} \|K_\mu f\|_{2,\mu}$.

Proof. i) When a semigroup has the ultracontractivity property, then it is hypercontractive. As the uniform hypercontractivity is equivalent to the uniform Sobolev logarithmic inequality, we conclude.

ii) Rothaus [20] has proved that if a measure satisfies a logarithmic Sobolev inequality with constant c , then it also satisfies a Poincaré inequality with constant $1/c$. Moreover, satisfying a Poincaré inequality is equivalent to the existence of a spectral gap. We easily bound $Z(\mu) \geq \int e^{-2\kappa(V+\beta)} d\gamma$.

From ii), we find the following estimate on the semigroup $(P_t^\mu)_{t \geq 0}$ (see Bakry [1]): there exists a uniform (in μ) constant $C > 0$ such that for all $t \geq 0$, $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, we get

$$\|P_t^\mu(K_\mu f)\|_{2,\mu} \leq e^{-t/C} \|K_\mu f\|_{2,\mu}.$$

□

3.3. The ω -limit set.

3.3.1. *Main results.* We will show that the time-changed process $\mu_{h(t)}$ (and not μ_t) is an asymptotic pseudotrajectory for Φ , where h is the deterministic time-change defined by

$$(3.6) \quad h(t) := r(e^t - 1) \forall t \geq 0.$$

It comes from the normalization of the occupation measure μ_t . The factor $(r+t)^{-1}$ disappears while considering

$$\frac{d}{dt} \mu_{h(t)} = \delta_{X_{h(t)}} - \mu_{h(t)}.$$

Theorem 3.6. *Under $\mathbb{P}_{x,r,\mu}$, the function $t \mapsto \mu_{h(t)}$ is almost surely an asymptotic pseudotrajectory for Φ .*

The proof is given in Section 6. This theorem enables us to describe the limit set of (μ_t) :

Corollary 3.7. *$\mathbb{P}_{x,r,\mu}$ -a.s., $\omega(\mu_t, t \geq 0)$ is weakly compact, invariant by Φ and attractor-free. It is also contained in the convex hull of the image of Π .*

An attractor-free set is a set that contains no (sub-)attractor (other than itself). The exact definition will be given later, in Section 6.

Theorem 3.8. *Assume that W is symmetric. Then, $\mathbb{P}_{x,r,\mu}$ -a.s., the ω -limit set of $(\mu_t, t \geq 0)$ is a connected subset of the fixed points of Π .*

The proof is given in Section 6. It immediately implies the following

Corollary 3.9. *Suppose that W is symmetric. If Π admits only finitely many fixed points, then $\mathbb{P}_{x,r,\mu}$ -a.s., $(\mu_t; t \geq 0)$ converges to one of them.*

3.3.2. *A sufficient condition for the global convergence.* For a symmetric W , we introduce the *free energy* (up to a multiplicative constant) corresponding to the ODE studied

$$(3.7) \quad \mathcal{F}(\mu) := \int_{\mathbb{R}^d} \log \left(\frac{d\mu}{d\gamma} \right) d\mu + \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x, y) \mu(dx) \mu(dy).$$

This functional is the sum of an entropy and an interacting energy term. The competition between them can imply the existence of a unique minimizer for \mathcal{F} (see [23]).

Theorem 3.10. *Suppose that W is symmetric and for all $y \in \mathbb{R}^d$, the function $x \mapsto V(x) + W(x, y)$ is strictly convex. Then there exists a unique probability measure μ_∞ such that $\lim_{t \rightarrow \infty} \mu_t = \mu_\infty$ $\mathbb{P}_{x,r,\mu}$ -a.s.*

Proof. Under our hypothesis, McCann has proved in [14] that \mathcal{F} has a unique critical point μ_∞ , which is a unique global minimum. It is also the unique fixed point of Π . So, $\lim_{t \rightarrow \infty} \mu_t = \mu_\infty$ $\mathbb{P}_{x,r,\mu}$ -a.s. □

4. STUDY OF THE DYNAMICAL SYSTEM Φ

4.1. Differentiating functions of probability measures. Here we endow the space $\mathcal{P}(\mathbb{R}^d; V)$ with a structure of infinite-dimensional differentiable manifold. This structure will be used only for differentiating functions defined on $\mathcal{P}(\mathbb{R}^d; V)$, which will be needed in the study of the semiflow and below in Section 5.

For any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ we consider the set $\mathcal{C}^p(\mu)$ ($p \geq 1$) of (germs of) curves defined on some neighborhood of zero $(-\varepsilon, \varepsilon)$ with values in $\mathcal{P}(\mathbb{R}^d; V)$, passing through μ at time zero and that are of class \mathcal{C}^p when they are considered as functions with values in the Banach space $\mathcal{M}(\mathbb{R}^d; V)$. Now we say that a function $\phi : \mathcal{P}(\mathbb{R}^d; V) \rightarrow \mathbb{R}$ is of class \mathcal{C}^p if for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ and any curve $f \in \mathcal{C}^p(\mu)$ the real function $\phi \circ f$ is of class \mathcal{C}^p . This will enable to define the differential of such a function ϕ . For any μ the tangent space at μ to $\mathcal{P}(\mathbb{R}^d; V)$ can be identified with the space $\mathcal{M}_0(\mathbb{R}^d; V)$ of zero-mass measures in $\mathcal{M}(\mathbb{R}^d; V)$. The differential is then the linear operator:

$$(4.1) \quad D\phi(\mu) \cdot \nu = \frac{d}{dt} \phi(\mu + t\nu)|_{t=0}, \quad \nu \in \mathcal{M}_0(\mathbb{R}^d; V).$$

The same definition can apply for functions with values in a Banach space or even in $\mathcal{P}(\mathbb{R}^d; V)$. As an example (to be used!), the maps $\mu \mapsto W * \mu(x)$ (for any point x) and Π (applying the Lebesgue theorem) are \mathcal{C}^∞ .

4.2. Existence of the semiflow. We first prove the local existence of the semiflow and then give sufficient conditions on the potentials for non-explosion. We recall the equation:

$$(4.2) \quad \Phi_t(\mu) = e^{-t}\mu + e^{-t} \int_0^t e^s \Pi(\Phi_s(\mu)) ds.$$

For proving the local existence of a solution, since $\mathcal{P}(\mathbb{R}^d; V)$ is not a vector space, we will proceed directly by approximation. The following lemma is helpful in order to find a good security cylinder.

Lemma 4.1. *For any $\beta > 1$, the application Π restricted to $\mathcal{P}_\beta(\mathbb{R}^d; V)$ is bounded and Lipschitz.*

Proof. First we need to show that $\mu \mapsto Z(\mu)$ is bounded from below. For $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, Lemma 2.1 asserts that $W * \mu(x) \leq 2\kappa\beta V(x)$. So we get:

$$Z(\mu) = \int_{\mathbb{R}^d} e^{-2W*\mu(x)} \gamma(dx) \geq \int_{\mathbb{R}^d} e^{-4\kappa\beta V(x)} \gamma(dx)$$

and thus we have the following bound for $\Pi(\mu)$:

$$(4.3) \quad \|\Pi(\mu)\|_V \leq \left(\int_{\mathbb{R}^d} e^{-4\kappa\beta V(x)} \gamma(dx) \right)^{-1} \int_{\mathbb{R}^d} V(x) \gamma(dx) =: C_\beta.$$

We know that Π is \mathcal{C}^∞ on $\mathcal{P}(\mathbb{R}^d; V)$ with the strong topology. Its differential (at μ) is the continuous linear operator $D\Pi(\mu) : \mathcal{M}_0(\mathbb{R}^d; V) \rightarrow \mathcal{M}_0(\mathbb{R}^d; V)$ defined by

$$(4.4) \quad D\Pi(\mu) \cdot \nu(dx) := -2 \left(W * \nu(x) - \int_{\mathbb{R}^d} W * \nu(y) \Pi(\mu)(dy) \right) \Pi(\mu)(dx).$$

Fix $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$. Since $|W * \nu(x)| \leq 2\kappa\|\nu\|_V V(x)$, we find that

$$\|D\Pi(\mu) \cdot \nu\|_V \leq 4\kappa(1 + C_\beta)\|\nu\|_V \int_{\mathbb{R}^d} V^2(x) \Pi(\mu)(dx).$$

But for $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, the same computation used for the bound of $\Pi(\mu)$ enables to control the last integral, hence we get a bound (call it C'_β) on the differential and Π is Lipschitz as stated. \square

Proposition 4.2. *For all $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ the ODE has a local solution. This defines a \mathcal{C}^∞ semiflow Φ for the strong topology.*

Proof. Let μ belong to $\mathcal{P}_\beta(\mathbb{R}^d; V)$ where we choose $\beta > 2\|\mu\|_V$. We introduce the classic Picard approximation scheme:

$$\begin{cases} \mu_t^{(0)} := \mu, \\ \mu_t^{(n)} := e^{-t}\mu + \int_0^t e^{s-t}\Pi\left(\mu_s^{(n-1)}\right) ds. \end{cases}$$

We set ε small enough such that $\|\mu\|_V + (1 - e^{-\varepsilon})C_\beta \leq \beta$ and $\varepsilon C'_\beta < 1$ where both constants were defined in Lemma 4.1. Then, for all n , $\mu_t^{(n)}$ is defined and belongs to $\mathcal{P}_\beta(\mathbb{R}^d; V)$, which makes $[0, \varepsilon) \times \mathcal{P}_\beta(\mathbb{R}^d; V)$ a good security cylinder. We have, for $t < \varepsilon$,

$$\|\mu_t^{(n+1)} - \mu_t^{(n)}\|_V \leq (1 - e^{-\varepsilon})C'_\beta \sup_{t < \varepsilon} \|\mu_t^{(n)} - \mu_t^{(n-1)}\|_V.$$

Now the series with general term $\sup_{t < \varepsilon} \|\mu_t^{(n+1)} - \mu_t^{(n)}\|_V$ converges and thus the sequence of functions $\mu^{(n)}$ is Cauchy for the topology of uniform convergence. Since $\mathcal{P}(\mathbb{R}^d; V)$ is complete, we have successfully built a solution on $[0, \varepsilon)$. There remains to show that the semiflow is smooth. We have seen that the map Π is \mathcal{C}^∞ for the strong topology. By induction, every Picard approximation $\mu \mapsto \mu_t^{(n)}$ is \mathcal{C}^∞ and it is enough to take the limit uniformly in μ on $\mathcal{P}_\beta(\mathbb{R}^d; V)$ to conclude. \square

Definition 4.3. A subset A of $\mathcal{P}(\mathbb{R}^d; V)$ is *positively invariant* (negatively invariant, invariant) for Φ provided $\Phi_t(A) \subset A$ ($A \subset \Phi_t(A)$, $\Phi_t(A) = A$) for all $t \geq 0$.

Proposition 4.4. Whenever that W is either symmetric or bounded in the second variable ($W(x, y) \leq \kappa V(x)$), then the semiflow Φ does not explode.

Proof. The case where $W(x, y)$ is bounded in y is easy. Since we have $W(x, y) \leq \kappa V(x)$, mimicking the proof of Lemma 4.1 enables to show that Π is globally bounded (call C the upper bound). This means that $\Phi_t(\mu)$ remains in the space $\mathcal{P}_C(\mathbb{R}^d; V)$, therefore it cannot explode.

Let us now assume that W is symmetric. We point out that the free energy (3.7) is not a Lyapunov function for (4.2) because in general the measure $\Phi_t(\mu)$ is not absolutely continuous with respect to the Lebesgue measure and so, $\mathcal{F}(\Phi_t(\mu)) = \infty$. Thus, consider the Lyapunov function $\mathcal{E}(\mu) := \mathcal{F}(\Pi(\mu))$. Indeed, \mathcal{F} restricted to absolutely continuous probability measures is a \mathcal{C}^∞ function for the strong topology (V -norm). We compute (thanks to the symmetry of W) for $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$

$$(4.5) \quad D\mathcal{F}(\mu) \cdot \nu = \int_{\mathbb{R}^d} \left[\log \left(\frac{d\mu}{d\gamma}(x) \right) + 2W * \mu(x) \right] d\nu(x).$$

But we recall that Π is \mathcal{C}^∞ and equation (4.4). So, since by composition $D\mathcal{E}(\mu) \cdot \nu = D\mathcal{F}(\Pi(\mu)) \circ D\Pi(\mu) \cdot \nu$, we obtain

$$D\mathcal{E}(\mu) \cdot \nu = -4 \int_{\mathbb{R}^d} (W * \Pi(\mu) - W * \mu) \left(W * \nu - \int_{\mathbb{R}^d} W * \nu d\Pi(\mu) \right) d\Pi(\mu).$$

It remains to choose $\nu = \Pi(\mu) - \mu$ in order to get

$$\frac{1}{4} \frac{d}{dt} \mathcal{E}(\Phi_t(\mu)) = - \int_{\mathbb{R}^d} (W * \nu)^2 d\Pi(\mu) + \left(\int_{\mathbb{R}^d} W * \nu d\Pi(\mu) \right)^2 \leq 0.$$

Therefore, for all $c > 0$, the sets $\{\mu; \mathcal{E}(\mu) \leq c\}$ are positively invariant. As they are (weakly) compact, the semiflow cannot explode. \square

4.3. An important set. We introduce here a crucial object for the analysis of the dynamical system Φ . Let

$$(4.6) \quad \text{Im}(\Pi) := \left\{ \Pi(\mu); \mu \in \mathcal{P}(\mathbb{R}^d; V) \right\},$$

and denote its convex hull by $\widehat{\text{Im}(\Pi)}$.

Proposition 4.5. $\widehat{\text{Im}(\Pi)}$ is a positively invariant set for the semiflow Φ which contains every negatively invariant bounded subset of $\mathcal{P}(\mathbb{R}^d; V)$.

Proof. To prove the result, it is enough to show for every $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ and every $t \geq 0$ the inequality

$$(4.7) \quad d_V(\Phi_t(\mu), \widehat{\text{Im}(\Pi)}) \leq e^{-t} d_V(\mu, \widehat{\text{Im}(\Pi)}),$$

where $d_V(\mu, X) := \inf\{\|\mu - \nu\|_V; \nu \in X\}$. But this inequality directly results from the Jensen inequality applied to the convex combination $\Phi_t(\mu) = e^{-t}\mu + e^{-t} \int_0^t e^s \Pi(\Phi_s(\mu)) ds$ and to the convex map $\mu \mapsto d_V(\mu, \widehat{\text{Im}(\Pi)})$. \square

4.4. Fixed points of Π . We show how the free energy functional \mathcal{F} (3.7) can help to find the fixed points of Π .

Proposition 4.6. Suppose that W is symmetric. Then the fixed points of Π are the minima of \mathcal{F} .

Proof. Equation (4.5) readily implies that, we have $D\mathcal{F}(\mu) \cdot \nu = 0$ for all $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$ if and only if $\mu = \Pi(\mu)$. So, the fixed points of Π are the critical points of \mathcal{F} . Moreover, \mathcal{F} is a convex functional. Indeed, it is a \mathcal{C}^∞ functional (on the set of absolutely continuous measures), with second differential $D^2\mathcal{F}(\mu)$. Let $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^d; V)$. We have:

$$D^2\mathcal{F}(\mu) \cdot (\nu_1, \nu_2) = \int_{\mathbb{R}^d} \nu_1(x) \nu_2(x) \mu(x)^{-1} \gamma(x) dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x, y) \nu_1(dx) \nu_2(dy)$$

and the convexity is a consequence of the nonnegativity of W . It then implies that $\mu = \Pi(\mu)$ is a minimum for \mathcal{F} . \square

Lemma 4.7. Whenever that W is either symmetric or bounded in the second variable, then the set of the fixed points of Π , $\{\mu \in \mathcal{P}(\mathbb{R}^d; V); \Pi(\mu) = \mu\}$, is a nonempty compact (for the weak topology) subset of $\mathcal{P}(\mathbb{R}^d; V)$.

Proof. Suppose first that W is symmetric. We will again use the free energy. Let $m := \inf\{\mathcal{E}(\mu); \mu \in \mathcal{P}(\mathbb{R}^d; V)\}$. There exists a sequence of probability measures (μ_n) absolutely continuous with respect to the Lebesgue measure such that $m \leq \mathcal{E}(\mu_n) \leq m + 1/n$. But, as proved in Proposition 4.4, for any $c > 0$, the set $\{\mu; \mathcal{E}(\mu) \leq c\}$ is compact. So, we extract a subsequence (μ_{n_k}) , converging (for the weak topology) to μ_∞ . As $\mu \mapsto W * \mu$ and $\mu \mapsto \Pi(\mu)$ are two continuous functions, $\mu \mapsto \mathcal{E}(\mu)$ is also weakly continuous and so $\mathcal{E}(\mu_\infty) = m$. We conclude by Proposition 4.6.

Suppose now that W is bounded in y : $W(x, y) \leq \kappa V(x)$. We have proved in Lemma 4.4 that $\Pi(\mu)$ maps (weakly) continuously the compact convex space $\mathcal{P}_C(\mathbb{R}^d; V)$ into itself. The Leray-Schauder fixed point theorem then ensures that the set $\{\mu \in \mathcal{P}(\mathbb{R}^d; V); \Pi(\mu) = \mu\}$ is nonempty. \square

5. STUDY OF THE FAMILY OF SEMIGROUPS $(P_t^\mu, t \geq 0, \mu \in \mathcal{P}_\beta(\mathbb{R}^d; V))$

In this section, we introduce two crucial functional inequalities for the family of semigroups P_t^μ : the spectral gap and the ultracontractivity. Since we consider these semigroups altogether for all the measures $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, we will prove that the constants involved in those properties are uniform in μ . The notion of ultracontractivity and its relation to the analysis of Markov semigroups were first studied by Davies and Simon [8] and recently by Röckner & Wang [19] for more general diffusions. The need for ultracontractivity will impose some kind of boundedness on the convolution term in the SDE that cannot be easily removed. Finally, thanks to these properties, we compute several estimates that prepare the proof of Section 6.

For any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, let as usual $L^2(\Pi(\mu))$ denote the Lebesgue space of Borel square-integrable functions with respect to the measure $\Pi(\mu)$. We remark that the space depends on μ , but we will

consider mainly the subspace $\mathcal{C}^0(\mathbb{R}^d; V) \subset L^2(\Pi(\mu))$. We denote

$$(f, g)_\mu := \int_{\mathbb{R}^d} f(x)g(x)\Pi(\mu)(dx)$$

the inner product on this space and $\|\cdot\|_{2,\mu}$ the associated norm. We introduce two operators : Q_μ is the "inverse" of A_μ , defined for any function f by

$$(5.1) \quad Q_\mu f := \int_0^\infty (P_t^\mu f - \Pi(\mu)f) dt$$

and K_μ is the orthogonal projector defined by

$$(5.2) \quad K_\mu f := f - \Pi(\mu)f.$$

They are linked together by the following relation $\forall f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$,

$$A_\mu \circ Q_\mu(f) = Q_\mu \circ A_\mu(f) = -K_\mu f.$$

Remark 5.1. *The integrability of $(P_t^\mu f - \Pi(\mu)f)$ will come from the uniform spectral gap obtained in Corollary 3.5.*

5.1. Uniform ultracontractivity. To prove that the family of semigroups $(P_t^\mu, t \geq 0, \mu \in \mathcal{P}_\beta(\mathbb{R}^d; V))$ is uniformly ultracontractive, we will rely on the following result of Röckner & Wang:

Lemma 5.2. ([19] corollary 2.5) *Let $(P_t, t \geq 0)$ be a Markov semigroup, with infinitesimal generator $A := \frac{1}{2}\Delta - (\nabla U, \nabla)$, and $\nabla^2 U \geq -K$. Assume that there exists a continuous increasing map $\chi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \setminus \{0\}$ such that*

- (1) $\lim_{r \rightarrow \infty} \frac{\chi(r)}{r} = \infty$,
- (2) *the mapping $g_\chi(r) := r\chi(m \log r)$ is convex on $[1, \infty)$ for any $m > 0$,*
- (3) $A|x|^2 \leq b - \chi(|x|^2)$ *for some $b > 0$.*

Then P_t has a unique invariant probability measure. If $\int_2^\infty \frac{dr}{r\chi(m \log r)} < \infty$, $m > 0$, then P_t is ultracontractive.

If moreover $\chi(r) = \chi r^\delta$, with $\chi > 0, \delta > 1$, then there exists $c = c(b, \chi) > 0$ such that for all $t \in (0, 1]$, $\|P_t\|_{2 \rightarrow \infty} \leq \exp(ct^{-\delta/(\delta-1)})$.

Proposition 3.4 . *The family of semigroups $(P_t^\mu, t \geq 0, \mu \in \mathcal{P}_\beta(\mathbb{R}^d; V))$ is uniformly ultracontractive: we have for all $1 \geq t > 0$ and $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$*

$$(5.3) \quad \|P_t^\mu\|_{2 \rightarrow \infty} := \sup_{f \in \mathcal{C}^\infty(\mathbb{R}^d; V) \setminus \{0\}} \frac{\|P_t^\mu f\|_\infty}{\|f\|_{2,\mu}} \leq \exp\left(ct^{-\delta/(\delta-1)}\right),$$

where $c > 0$ is independent from μ .

Proof. We apply Lemma 5.2 with $U := V + W * \mu$ and find that each $(P_t^\mu)_{t \geq 0}$ is ultracontractive. Indeed, the conditions (1.7) and the growth condition on V , all together imply that there exist $a, b > 0$ such that for $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$

$$A_\mu |x|^2 = d - 2(\nabla W * \mu(x), x) - 2(\nabla V(x), x) \leq b - a|x|^{2\delta}.$$

As we let $\chi(r) := r^\delta$, with $\delta > 1$, we find that the constant c is uniform in μ . Thus, we have the uniform ultracontractivity. \square

We recall that, as a consequence of Proposition 3.4, there exists $C > 0$, uniform in μ , such that $\forall t \geq 0$,

$$\|P_t^\mu(K_\mu f)\|_{2,\mu} \leq e^{-t/C} \|K_\mu f\|_{2,\mu}.$$

We are now able to derive some usefull bounds on the operator Q_μ .

Proposition 5.3. *For all $\varepsilon > 0$, there exists a positive constant $K(\varepsilon)$ such that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, $x \in \mathbb{R}^d$, $f \in \mathcal{C}^0(\mathbb{R}^d; V)$:*

$$(5.4) \quad |Q_\mu f(x)| \leq (\varepsilon V(x) + K(\varepsilon)) \|f\|_V.$$

Proof. Let $t_0 \in (0, 1]$ (we will choose it precisely later). We have:

$$|Q_\mu f(x)| \leq \int_0^\infty |P_t^\mu(K_\mu f)(x)| dt = \int_0^{t_0} |P_t^\mu(K_\mu f)(x)| dt + \int_{t_0}^\infty |P_t^\mu(K_\mu f)(x)| dt.$$

We begin to work with the second right-hand term. By use of the uniform ultracontractivity and the uniform spectral gap, we have

$$\begin{aligned} \int_{t_0}^\infty |P_t^\mu(K_\mu f)(x)| dt &= \int_0^\infty |P_{t_0}^\mu P_t^\mu(K_\mu f)(x)| dt, \\ \int_{t_0}^\infty |P_t^\mu(K_\mu f)(x)| dt &\leq \exp\left(ct_0^{-\delta/(\delta-1)}\right) \int_0^\infty e^{-t/C_1} dt \|K_\mu f\|_{2,\mu}. \end{aligned}$$

As K_μ is an orthogonal projector, $\|K_\mu f\|_{2,\mu} \leq \|f\|_{2,\mu} \leq \left(\int V^2 d\Pi(\mu)\right)^{1/2} \|f\|_V$, and we get

$$\int_{t_0}^\infty |P_t^\mu(K_\mu f)(x)| dt \leq C_1 \exp\left(ct_0^{-\delta/(\delta-1)}\right) \left(\int V^2 d\Pi(\mu)\right)^{1/2} \|f\|_V.$$

We now have to work with the first right-hand term. We have with the naive estimate (2.8):

$$|P_t^\mu f(x)| \leq \|f\|_V P_t^\mu V(x) \leq \|f\|_V \mathbb{E} \mathcal{E}_\mu(X_t^\mu) \leq \mathcal{E}_\mu(x) e^{Dt} \|f\|_V.$$

Since $\mathcal{E}_\mu(x) \leq 3\kappa\beta V(x)$, we finally find

$$\int_0^{t_0} |P_t^\mu K_\mu f(x)| dt \leq 4\kappa\beta \int_0^{t_0} e^{Dt} dt \|f\|_V V(x).$$

Now we choose t_0 small enough such that $4\kappa\beta \int_0^{t_0} e^{Dt} dt \leq \varepsilon$ to conclude. \square

Proposition 5.4. *For all $\varepsilon > 0$, there exists $K_1(\varepsilon) > 0$ such that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, $x \in \mathbb{R}^d$, $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, we have $Q_\mu f \in \mathcal{C}^1(\mathbb{R}^d)$ and:*

$$(5.5) \quad |\nabla Q_\mu f(x)| \leq (\varepsilon V(x) + K_1(\varepsilon)) \|f\|_V.$$

Proof. Suppose that f is smooth. We introduce two operators: the ‘carré du champ’ $\Gamma(f) = |\nabla f|^2$ and $\Gamma_2^\mu(f) = |\nabla^2 f|^2 + (\nabla f, \nabla^2(V + W * \mu) \nabla f)$. As we have the curvature condition (1.7), we get (for the curvature $M \in \mathbb{R}$) $\Gamma_2^\mu(f) \geq M\Gamma(f)$. The Γ_2 -criterion implies the following (see Ledoux [13] p22), $\forall f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, $\forall t > 0$

$$(5.6) \quad |\nabla P_t^\mu(K_\mu f)|^2 \leq \frac{M}{e^{2Mt} - 1} |P_t^\mu(K_\mu f)|^2.$$

Indeed, one can show that

$$\begin{aligned} \int_{t_0}^\infty |\nabla P_t^\mu(K_\mu f)(x)| dt &\leq \sqrt{\frac{M}{e^{2Mt_0} - 1}} \int_0^\infty [P_{t_0}(P_t^\mu(K_\mu f))^2(x)]^{1/2} dt \\ &\leq C(t_0) \|f\|_V \left(\int V^4 d\Pi(\mu)\right)^{1/4} \end{aligned}$$

where $C(t_0) = 2C_1 \sqrt{\frac{M}{e^{2Mt_0} - 1}} \exp\{ct_0^{-\delta/(\delta-1)}/2\}$. Finally, similarly to Proposition 2.2, one proves that

$$\mathbb{E}_x V^2(X_t^\mu) \leq \mathcal{E}_\mu^2(x) e^{2Dt} \leq (3\kappa\beta)^2 V^2(x) e^{2Dt}.$$

\square

5.2. Regularity with respect to the measure μ . First, consider the Banach space \mathcal{B} of bounded linear operators from $\mathcal{C}^\infty(\mathbb{R}^d; V) \subset L^2(\gamma)$, endowed with the norm $\|f\|_{2,\mu,1} := \|f\|_{2,\mu} + \|A_\mu f\|_{2,\mu}$, to the same space equipped with the standard quadratic norm. We endow \mathcal{B} with the operator norm. Then, A_μ obviously belongs to the closed subset of \mathcal{B} consisting in operators A such that $A1 = 0$. This allows us to state and prove the following:

Proposition 5.5. *The mappings $\mu \mapsto A_\mu$ and $\mu \mapsto K_\mu$ are \mathcal{C}^∞ . For any function $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, the application $\mu \mapsto Q_\mu f$ is \mathcal{C}^∞ for the strong topology and we have for the differentials (for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$):*

$$\begin{aligned} D(A_\mu f) \cdot \nu &= -(\nabla W * \nu, \nabla f); \\ D(K_\mu f) \cdot \nu &= -(D\Pi(\mu) \cdot \nu)(f); \\ D(Q_\mu f) \cdot \nu &= (D\Pi(\mu) \cdot \nu)(Q_\mu f) + Q_\mu(\nabla W * \nu, \nabla Q_\mu f). \end{aligned}$$

Proof. We already know that $\mu \mapsto W * \mu$ and Π are \mathcal{C}^∞ ; so there is nothing to prove in case of A_μ or K_μ . To look at Q_μ , we need to consider the resolvent operator of P_t^μ :

$$(5.7) \quad R_\lambda^\mu := \int_0^\infty e^{-\lambda t} P_t^\mu dt = (\lambda - A_\mu)^{-1}, \quad \forall \lambda > 0.$$

For $\lambda > 0$, we define the following approximation of Q_μ

$$(5.8) \quad Q_\mu(\lambda) := \int_0^\infty e^{-\lambda t} P_t^\mu K_\mu dt = K_\mu(\lambda - A_\mu)^{-1}.$$

As $\mu \mapsto K_\mu$ and $\mu \mapsto A_\mu$ are \mathcal{C}^∞ , we find by composition that the map $\mu \mapsto R_\lambda^\mu f$ is \mathcal{C}^∞ .

Now let $\beta > 1$ and consider the measures in $\mathcal{P}_\beta(\mathbb{R}^d; V)$. The uniform spectral gap shows that there exist $C, C_1 > 0$ such that we have

$$\|Q_\mu f - Q_\mu(\lambda) f\|_V \leq \int_0^\infty dt (1 - e^{-\lambda t}) \|P_t^\mu K_\mu f\|_V \leq \lambda C \|f\|_V \int_0^\infty t e^{-t C_1} dt.$$

Hence the convergence of $Q_\mu(\lambda)$ towards Q_μ is uniform with respect to μ on $\mathcal{P}_\beta(\mathbb{R}^d; V)$. As a by-product, $\mu \mapsto Q_\mu f$ is continuous.

We have the following differential:

$$\begin{aligned} DQ_\mu(\lambda) \cdot \nu &= (DK_\mu \cdot \nu)(\lambda - A_\mu)^{-1} + K_\mu(\lambda - A_\mu)^{-1} (DA_\mu \cdot \nu)(\lambda - A_\mu)^{-1} \\ &= (D\Pi(\mu) \cdot \nu)(\lambda - A_\mu)^{-1} + K_\mu(\lambda - A_\mu)^{-1} (\nabla W * \nu, \nabla)(\lambda - A_\mu)^{-1}. \end{aligned}$$

We will prove that each right side term of the preceding equality converges uniformly. For the first term, we have for all $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$:

$$(D\Pi(\mu) \cdot \nu)((\lambda - A_\mu)^{-1} f) = (D\Pi(\mu) \cdot \nu)(K_\mu(\lambda - A_\mu)^{-1} f)$$

and therefore

$$\lim_{\lambda \rightarrow 0+} (D\Pi(\mu) \cdot \nu)((\lambda - A_\mu)^{-1} f) = (D\Pi(\mu) \cdot \nu)(Q_\mu f)$$

where the convergence is uniform in μ . It remains to prove the convergence of the second term. We have

$$K_\mu(\lambda - A_\mu)^{-1} (\nabla W * \nu, \nabla)((\lambda - A_\mu)^{-1} f) = Q_\mu(\lambda) (\nabla W * \nu, \nabla Q_\mu(\lambda) f).$$

If we manage to prove that $\nabla Q_\mu(\lambda) f$ converges (uniformly in μ) to $\nabla Q_\mu f$, then we are done. We have by definition of $Q_\mu(\lambda)$:

$$\nabla Q_\mu(\lambda) f = \int_0^\infty \nabla(P_t^\mu K_\mu f) e^{-\lambda t} dt$$

and therefore

$$|\nabla Q_\mu f - \nabla Q_\mu(\lambda) f| \leq \int_0^\infty |\nabla(P_t^\mu K_\mu f)| (1 - e^{-\lambda t}) dt.$$

We use inequality (5.6) to prove that this family of differentials converges uniformly with respect to μ ; so $\mu \mapsto Q_\mu f$ is actually \mathcal{C}^1 with the differential given in the statement of the proposition. \square

Remark 5.6. *Looking at the differential $D(Q_\mu f)$, we see that it is itself a \mathcal{C}^1 function of μ , so by induction it can be proved that $\mu \mapsto Q_\mu f$ is \mathcal{C}^∞ . We could have also proved that $\mu \mapsto P_t^\mu f$ is \mathcal{C}^∞ . But these results will not be needed in the remainder.*

Corollary 5.7. *For every $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, we have the uniform inequality*

$$|(DQ_\mu \cdot \nu)(f)(x)| \leq (\varepsilon V(x)^2 + K_2(\varepsilon)) \|f\|_V \|\nu\|_V.$$

Proof. We have the following:

$$|(DQ_\mu \cdot \nu)(f)(x)| \leq |(D\Pi(\mu) \cdot \nu)(Q_\mu f)| + |Q_\mu(\nabla W * \nu(x), \nabla Q_\mu f(x))|.$$

We will treat each of the two terms on the right side separately. If we consider the second right hand term, we find

$$\begin{aligned} |Q_\mu(\nabla W * \nu(x), \nabla Q_\mu f(x))| &\leq (\varepsilon V^2(x) + K(\varepsilon)) \|(\nabla W * \nu, \nabla Q_\mu f)\|_{V^2} \\ &\leq (\varepsilon V^2(x) + K(\varepsilon)) \|\nabla W * \nu\|_V \|\nabla Q_\mu f\|_V \\ &\leq (\varepsilon V^2(x) + K'(\varepsilon)) \|\nu\|_V \|f\|_V. \end{aligned}$$

We work now with the other member of the inequality.

$$\begin{aligned} |(D\Pi(\mu) \cdot \nu)(Q_\mu f)| &\leq 2 \int |Q_\mu f(x)| \left| W * \nu(x) - \int W * \nu d\Pi(\mu) \right| \Pi(\mu)(dx) \\ &\leq C \|f\|_V \|\nu\|_V \int (\varepsilon V(x) + K(\varepsilon)) (V(x) + 1) \Pi(\mu)(dx) \\ &\leq C' \|f\|_V \|\nu\|_V. \end{aligned}$$

Putting the pieces together, we are done. \square

6. BEHAVIOR OF THE OCCUPATION MEASURE

6.1. Tightness of $(\mu_t, t \geq 0)$. Thanks to the potential V , we manage to obtain the weaker form of compactness of the occupation measure: tightness.

Proposition 3.3 . *Let x, r, μ be given. Then there exists $\beta > 1$ such that $\mathbb{P}_{x,r,\mu}$ -a.s. $\mu_t \in \mathcal{P}_\beta(\mathbb{R}^d; V)$ for all t large enough.*

Proof. We set $\phi(t) := \int_0^t V(X_s) ds$. All we need to prove is that $\mathbb{P}_{x,r,\mu}$ -a.s. $\phi(t) = O(t)$. We use again the Lyapunov functional $\mathcal{E}_\mu(x) = V(x) + W * \mu(x)$. We have already shown:

$$\begin{aligned} \mathcal{E}_{\mu_t}(X_t) &= \mathcal{E}_\mu(x) + \int_0^t (\nabla \mathcal{E}_{\mu_s}(X_s), dB_s) - \int_0^t |\nabla \mathcal{E}_{\mu_s}(X_s)|^2 ds \\ &\quad + \frac{1}{2} \int_0^t \Delta \mathcal{E}_{\mu_s}(X_s) ds + \int_0^t (W(X_s, X_s) - W * \mu_s(X_s)) \frac{ds}{r+s}. \end{aligned}$$

The strong law of large numbers (for martingales) implies that a.s. for t large enough we will have $\int_0^t (\nabla \mathcal{E}_{\mu_s}(X_s), dB_s) \leq \frac{1}{2} \int_0^t |\nabla \mathcal{E}_{\mu_s}(X_s)|^2 ds$, and therefore we get the a.s. inequality for t large enough:

$$\int_0^t |\nabla \mathcal{E}_{\mu_s}(X_s)|^2 ds \leq 2\mathcal{E}_\mu(x) + \int_0^t \Delta \mathcal{E}_{\mu_s}(X_s) ds + \frac{2}{r} \int_0^t W(X_s, X_s) ds.$$

Now we want to find an integral inequality on ϕ . To this aim, we will control separately each of the three terms of the last inequality and let $\phi(t)$ appear.

i) From the growth assumption on V , for any $\epsilon > 0$ we can find $k_\epsilon > 0$ such that $V \leq k_\epsilon + \epsilon |\nabla V|^2$, thus by means of an integration we get a.s.

$$\phi(t) \leq k_\epsilon t + \epsilon \int_0^t |\nabla \mathcal{E}_{\mu_s}(X_s)|^2 ds.$$

ii) From the domination condition (1.6) on W , we have $\Delta W * \mu(x) \leq \kappa(V(x) + \mu(V))$; the growth condition (1.8) on V ensures the inequality $\Delta V \leq aV$ for some $a > 0$; therefore we get a.s.

$$\Delta \mathcal{E}_{\mu_s}(X_s) \leq \kappa \mu(V) + (\kappa + a)V(X_s) + \frac{\kappa}{r+s} \phi(s).$$

iii) The domination condition (1.6) leads also to

$$W(X_s, X_s) \leq 2\kappa V(X_s).$$

Putting all the pieces together, we find the following inequality (denoting by C_1 a deterministic positive constant) for t large enough

$$\phi(t) \leq k_\epsilon t + \epsilon \left((a + 2\kappa)\phi(t) + \kappa \int_0^t \frac{ds}{r+s} \phi(s) + C_1 t + \frac{4\kappa}{r} \phi(t) \right).$$

Now we choose ϵ small enough such that $\epsilon(a + \kappa(2 + \frac{4}{r})) < 1$ and so, we have:

$$\phi(t) \leq C'_1 t + \int_0^t \frac{C'_2 ds}{r+s} \phi(s),$$

with C'_i another positive constants. Finally, thanks to Gronwall's lemma, there exists a positive deterministic constant β such that $\phi(t) \leq \beta t$ for t large enough as required (because $C'_2 < 1$). \square

Proposition 6.1. *Let $\beta > 1$ such that $\mu_t \in \mathcal{P}_\beta(\mathbb{R}^d; V)$ for t large enough. For all $n \in \mathbb{N}$, we have that $\mathbb{E}_{x,r,\mu}(V^n(X_t))$ is bounded.*

Proof. We drop the subscripts x, r, μ in the following. We will prove the result for the Lyapunov function $\mathcal{E}_\mu(x)$ instead of V . Let $n = 1$. We apply the Itô formula to $(s, x) \mapsto \mathcal{E}_{\mu_s}(x)$:

$$\begin{aligned} d\mathcal{E}_{\mu_s}(X_s) &= (\nabla \mathcal{E}_{\mu_s}(X_s), dB_s) - |\nabla \mathcal{E}_{\mu_s}(X_s)|^2 ds + \frac{1}{2} \Delta \mathcal{E}_{\mu_s}(X_s) ds \\ &+ (W(X_s, X_s) - W * \mu_s(X_s)) \frac{ds}{r+s}. \end{aligned}$$

The condition (1.7) on $V + W * \mu$ (uniform in μ) leads to (for t large enough):

$$\forall \alpha > 0, \exists K_\alpha = K(\alpha, \beta, V); \mathcal{E}_{\mu_t}(X_t) \leq \alpha |\nabla \mathcal{E}_{\mu_t}(X_t)|^2 + K_\alpha.$$

From the domination condition (1.6) on W and the growth condition on V , there exists $a > 0$ such that $\Delta \mathcal{E}_{\mu_t}(X_t) \leq a \mathcal{E}_{\mu_t}(X_t)$. These bounds lead, for all $t \geq s$ large enough, to

$$\begin{aligned} \mathbb{E} \mathcal{E}_{\mu_t}(X_t) &\leq \mathbb{E} \mathcal{E}_{\mu_s}(X_s) + \frac{1}{2\alpha} \int_s^t (K_\alpha - \mathbb{E} \mathcal{E}_{\mu_u}(X_u)) du + \frac{a}{2} \int_s^t \mathbb{E} \mathcal{E}_{\mu_u}(X_u) du \\ &+ \kappa \int_s^t \mathbb{E} V(X_u) (r+u)^{-1} du. \end{aligned}$$

Now, we can choose α such that $1/\alpha - a = 2a$ and we recall that $V(X_t) = O(t)$. Therefore the preceding inequality becomes with $M = M(\beta, V)$

$$\mathbb{E} \mathcal{E}_{\mu_t}(X_t) \leq \mathbb{E} \mathcal{E}_{\mu_s}(X_s) - a \int_s^t \mathbb{E} \mathcal{E}_{\mu_u}(X_u) du + M(t-s)$$

We divide both sides by $t - s$ and let $s \rightarrow t$. Let $x(t) := \mathbb{E}\mathcal{E}_{\mu_t}(X_t)$. Solving the preceding inequality boil down to solve $\dot{x} \leq M - ax$. The solution satisfies $x(t) \leq \left(x(0) + M \int_0^t e^{as} ds\right) e^{-at}$ and we finally obtain:

$$\mathbb{E}\mathcal{E}_{\mu_t}(X_t) \leq KV(x)e^{-at} + \frac{M}{a}(1 - e^{-at}).$$

We conclude the general case $n \geq 1$ by induction. \square

6.2. Asymptotic behavior. We define the family of measures $\{\varepsilon_{t,t+s}; t \geq 0, s \geq 0\}$ by

$$(6.1) \quad \varepsilon_{t,t+s} := \int_t^{t+s} (\delta_{X_{h(u)}} - \Pi(\mu_{h(u)})) du.$$

This family will play an important role: it will be essential for proving that $t \mapsto \mu_{h(t)}$ is an asymptotic pseudotrajectory for Φ .

Proposition 6.2. *i) Let t be large enough. For all $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$ and every $T > 0$ there exists a positive constant $K = K(V, W, x)$ such that for all $\delta > 0$*

$$\mathbb{P}_{x,r,\mu} \left(\sup_{0 \leq s \leq T} |\varepsilon_{t,t+s} f| > \delta \right) \leq K \delta^{-2} e^{-t} \|f\|_V^2.$$

ii) For all $T > 0$ and all $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, we have $\mathbb{P}_{x,r,\mu}$ -a.s.

$$\lim_{t \rightarrow \infty} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s} f| = 0.$$

Proof. i) We need the uniform estimates on the family of semigroups (P_t^μ) proved in Section 5. Let $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$. We begin to rewrite

$$\varepsilon_{t,t+s} f = \int_{h(t)}^{h(t+s)} A_{\mu_u} Q_{\mu_u} f \frac{du}{r+u}.$$

We consider the \mathcal{C}^2 -valued process $(t, x) \mapsto Q_{\mu_{h(t)}} f(x)$, which is of class \mathcal{C}^2 and a \mathcal{C}^1 -semimartingale. Indeed it is easy to see that $t \mapsto \mu_{h(t)}$ is a.s. a bounded variation process with values in $\mathcal{M}(\mathbb{R}^d; V)$. Since Proposition 5.4 shows that $\mu \mapsto Q_\mu f$ is also \mathcal{C}^1 , the claim follows by composition. So, we apply the generalized Itô formula to $(t, x) \mapsto h(t)^{-1} Q_{\mu_{h(t)}} f(x)$ and decompose $\varepsilon_{t,t+s}$ in four parts (and we will control each term separately):

$$\varepsilon_{t,t+s} f = \varepsilon_{t,t+s}^{(1)} f + \varepsilon_{t,t+s}^{(2)} f + \varepsilon_{t,t+s}^{(3)} f + \varepsilon_{t,t+s}^{(4)} f$$

with

$$\begin{aligned} \varepsilon_{t,t+s}^{(1)} f &= -\frac{1}{h(t+s)} Q_{\mu_{h(t+s)}} f(X_{h(t+s)}) + \frac{1}{h(t)} Q_{\mu_{h(t)}} f(X_{h(t)}) \\ \varepsilon_{t,t+s}^{(2)} f &= -\int_{h(t)}^{h(t+s)} Q_{\mu_u} f(X_u) \frac{du}{(r+u)^2} \\ \varepsilon_{t,t+s}^{(3)} f &= \int_{h(t)}^{h(t+s)} \frac{\partial}{\partial u} Q_{\mu_u} f(X_u) \frac{du}{r+u} \\ \varepsilon_{t,t+s}^{(4)} f &= M_{h(t+s)}^f - M_{h(t)}^f \end{aligned}$$

where M_t^f is the local martingale $M_t^f := \int_0^t \nabla Q_{\mu_u} f(X_u) \frac{dB_u}{r+u}$.

We recall the estimates of Propositions 5.3 and 5.4: $\forall \varepsilon > 0, f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$,

$$\begin{aligned} |Q_{\mu_{h(t)}} f(X_{h(t)})| &\leq \|f\|_V (\varepsilon V(X_{h(t)}) + K(\varepsilon)) \\ |\nabla Q_{\mu_{h(t)}} f(X_{h(t)})| &\leq \|f\|_V (\varepsilon V(X_{h(t)}) + K_1(\varepsilon)). \end{aligned}$$

We also remind that $\int_0^t V(X_s)ds = O(t)$ and $V(X_t) = O(t)$ a.s. Now, we are able to control each part of $\varepsilon_{t,t+s}$ and find for all $\varepsilon > 0$ and t large enough:

$$\begin{aligned} |\varepsilon_{t,t+s}^{(1)} f| &\leq h(t)^{-1} (|Q_{\mu_{h(t+s)}} f(X_{h(t+s)})| + |Q_{\mu_{h(t)}} f(X_{h(t)})|) \\ &\leq h(t)^{-1} \|f\|_V (\varepsilon(V(X_{h(t+s)}) + V(X_{h(t)})) + 2K(\varepsilon)) \end{aligned}$$

so $\sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(1)} f| \leq C_1 h(t)^{-1} \|f\|_V$ a.s.; and similarly

$$|\varepsilon_{t,t+s}^{(2)} f| \leq \int_{h(t)}^{h(t+s)} (\varepsilon V(X_u) + K(\varepsilon)) \frac{du}{(r+u)^2} \|f\|_V \leq \frac{C_2 \|f\|_V}{h(t)^2} \int_{h(t)}^{h(t+s)} V(X_u) du$$

so that $\sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(2)} f| \leq C_2 h(t)^{-1} \|f\|_V$ a.s.

For the third part of $\varepsilon_{t,t+s}$, we will use Markov's inequality and the bound on the differential of Q_μ given in Corollary 5.7:

$$\begin{aligned} \mathbb{P} \left(\sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(3)} f| \geq \delta \right) &\leq \delta^{-2} \int_{h(t)}^{h(t+T)} \mathbb{E} |(DQ_{\mu_u} \cdot \dot{\mu}_u)(f)(X_u)|^2 \frac{du}{r+u} \\ &\leq \frac{C}{\delta^2} \|f\|_V^2 \int_{h(t)}^{h(t+T)} \mathbb{E} (V^6(X_u)) \frac{du}{(r+u)^3}. \end{aligned}$$

Recall, that we have proved that for all $\varepsilon > 0$, $n \in \mathbb{N}$ and t large enough, we obtain $\mathbb{E}[V^n(X_t)] = o(t^\varepsilon)$. Then, there exists some (uniform) constant C_3 such that

$$\mathbb{P} \left(\sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(3)} f| \geq \delta \right) \leq \frac{C_3}{\delta^2} h(t)^{-1} \|f\|_V^2.$$

Since the quadratic variation of $M_{h(t+s)}^f - M_{h(t)}^f$ is bounded by the quantity $\|f\|_V^2 \int_{h(t)}^{h(t+T)} (\varepsilon V(X_u) + K_1(\varepsilon))^2 \frac{du}{(r+u)^2}$, Burkholder-Davis-Gundy's inequality implies directly

$$(6.2) \quad \mathbb{P}_{x,r,\mu} \left(\sup_{s \in [0,T]} |\varepsilon_{t,t+s}^{(4)} f| \geq \delta \right) \leq \frac{C_4}{\delta^2} h(t)^{-1} \|f\|_V^2.$$

ii) Let $T > 0$ and $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$. We just need to prove that

$$\lim_{t \rightarrow \infty} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(4)} f| = \lim_{t \rightarrow \infty} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(3)} f| = 0.$$

We will use Borel-Cantelli's lemma. First, for all $\varepsilon > 0$, we have by Doob's inequality added to Burkholder-Davis-Gundy's inequality that

$$\mathbb{P}_{x,r,\mu} \left(\sup_{n \leq t < n+1} \sup_{s \in [0,T]} |\varepsilon_{t,t+s}^{(4)} f| \geq \delta \right) \leq \frac{C}{\delta^2} \|f\|_V^2 \sup_{n \leq t < n+1} (\varepsilon + h(t)^{-1}).$$

As it is true for all $\varepsilon > 0$, we deduce from the preceding inequality that

$$\mathbb{P}_{x,r,\mu} \left(\sup_{n \leq t < n+1} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(4)} f| \geq \delta \right) \leq \frac{C}{\delta^2} \|f\|_V^2 h(n)^{-1}.$$

As we know that $\sum_n h(n)^{-1}$ converges, we conclude by Borel-Cantelli's lemma that a.s.

$$\lim_{n \rightarrow \infty} \sup_{n \leq t < n+1} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(4)} f| = 0.$$

The same argument for $|\varepsilon_{t,t+s}^{(3)} f|$ permits to conclude. \square

Lemma 6.3. ([4]) *If for all $T > 0$, all $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, we have*

$$\lim_{t \rightarrow \infty} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s} f| = 0 \text{ a.s.,}$$

then the time-changed process, given by $\mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R}^d; V)$, $t \mapsto \mu_{h(t)}$ is a.s. an asymptotic pseudotrajectory for Φ (for the weak topology of measures).

Proof. We recall that the family $(\mu_t, t \geq 0)$ is a.s. tight and by Prokhorov's theorem, as we work in a Polish space, it is equivalent to the relative compactness of $(\mu_t, t \geq 0)$. Let $\mathcal{C}(\mathbb{R}, \mathcal{P}(\mathbb{R}^d; V))$ be the space of continuous paths $\nu : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}^d; V)$ equipped with the weak topology. Let θ be the translation flow $\theta : \mathcal{C}(\mathbb{R}, \mathcal{P}(\mathbb{R}^d; V)) \times \mathbb{R} \rightarrow \mathcal{C}(\mathbb{R}, \mathcal{P}(\mathbb{R}^d; V)); \theta_t(\nu)(s) = \nu(t+s)$ and $\hat{\Phi}$ be the mapping $\hat{\Phi} : \mathcal{C}(\mathbb{R}, \mathcal{P}(\mathbb{R}^d; V)) \rightarrow \mathcal{C}(\mathbb{R}, \mathcal{P}(\mathbb{R}^d; V)); \hat{\Phi}(\nu)(t) = \Phi_t(\nu(0))$. Benaïm [2] (theorem 3.2) asserts that a continuous map $\nu : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R}^d; V)$ is an asymptotic pseudotrajectory for the semiflow Φ if and only if ν is uniformly continuous (for the weak topology) and every limit point of $\{\theta_t(\nu); t \geq 0\}$ is a fixed point for $\hat{\Phi}$. We begin to prove that $\mu_{h(t)}$ is uniformly continuous for the weak topology. We have by definition of μ_t that

$$|\mu_{h(t+s)} f - \mu_{h(t)} f| \leq \int_t^{t+s} (|\mu_{h(u)} f| + |f(X_{h(u)})|) du.$$

As $\int_0^t V(X_u) du = O(t)$ a.s., this enables us to show that for all t large enough

$$(6.3) \quad |\mu_{h(t+s)} f - \mu_{h(t)} f| \leq 2\beta s \|f\|_V.$$

We put these estimates in equation (2.16) and the uniform continuity follows.

Let $I_F : C^0(\mathbb{R}, \mathcal{P}(\mathbb{R}^d; V)) \rightarrow C^0(\mathbb{R}, \mathcal{M}(\mathbb{R}^d; V))$ be the mapping defined by

$$I_F(\nu)(t) := \nu(0) + \int_0^t F(\nu(s)) ds$$

where F is the vector field $F(\mu) = \Pi(\mu) - \mu$. Then, by definition of $\mu_{h(t)}$

$$\theta_t(\mu_{h(\cdot)}) = I_F(\theta_t(\mu_{h(\cdot)})) + \varepsilon_{t,t+\cdot}.$$

Thus, by relative compactness of $(\mu_{h(t)}, t \geq 0)$ and continuity of I_F , we find that $\lim_{t \rightarrow \infty} \varepsilon_{t,t+\cdot} = 0$ in $C^0(\mathbb{R}, \mathcal{M}(\mathbb{R}^d; V))$ if and only if every limit point η of $(\theta_t(\mu_{h(\cdot)}))$ satisfies $\eta = I_F(\eta)$, that is $\eta = \hat{\Phi}(\eta)$. \square

Theorem 3.6 . $\mathbb{P}_{x,r,\mu}$ -a.s., the function $t \mapsto \mu_{h(t)}$ is an asymptotic pseudotrajectory for Φ .

Proof. It suffices to combine Proposition 6.2 with Lemma 6.3. \square

6.3. Back to the dynamical system: a global attractor for the semiflow. We have defined in Section 4 the smooth dynamical system Φ , with respect to the strong topology. But, in order to study the asymptotic behavior of $(\mu_t, t \geq 0)$, it will prove technically easier to work with the weak topology. Indeed, a good candidate to be an attractor of the semiflow is the ω -limit set of (μ_t) for the weak topology

$$(6.4) \quad \omega(\mu_t, t \geq 0) := \bigcap_{t \geq 0} \overline{\{\mu_s; s \geq t\}}$$

which is (a.s.) weakly compact, since it is contained in $\mathcal{P}_\beta(\mathbb{R}^d; V)$ a.s. Therefore, we will regard for now on the semiflow Φ with the weak topology:

Proposition 6.4. $\Phi : \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d; V)$ induces a continuous semiflow with respect to the weak topology.

Proof. Since $\mu \mapsto W * \mu(x)$ is readily weakly continuous (see the domination condition again), we see that Π is weakly continuous. Now, going back to the Picard approximation scheme of Section 4, it results that $\mu \mapsto \mu_t^{(n)}$ is weakly continuous for every n and t . Passing to the limit, we are done. \square

Now we need to recall a short list of important definitions coming from the theory of dynamical systems.

Definition 6.5. *a) A subset A of $\mathcal{P}(\mathbb{R}^d; V)$ is an attracting set (respectively attractor) for Φ provided:*

- (1) *A is nonempty, compact for the weak topology and positively invariant, (respectively invariant) and*
- (2) *A has a neighborhood $\mathcal{N} \subset \mathcal{P}(\mathbb{R}^d; V)$ such that $d(\Phi_t(\mu), A) \rightarrow 0$ as $t \rightarrow +\infty$ uniformly in $\mu \in \mathcal{N}$.*

b) The basin of attraction of an attractor $K \subset A$ for $\Phi|_A = (\Phi_t|_A)_t$ is the positively invariant open set (in A) comprising all points whose orbits asymptotically are in K . That is

$$B(K, \Phi|_A) := \{\mu \in A; \lim_{t \rightarrow \infty} d(\Phi_t(\mu), K) = 0\}.$$

c) A global attracting set (respectively global attractor) is an attracting set (respectively attractor) whose basin is the whole space $\mathcal{P}(\mathbb{R}^d; V)$.

d) Let A be a positively invariant set for Φ . An attractor for $\Phi|_A$ is proper if it is different from A .

e) An attractor-free set is a nonempty compact invariant set A such that $\Phi|_A$ has no proper attractor.

Our aim is now to describe the limit set of μ_t and find a global attracting set for Φ . The natural candidate is the limit set $\omega(\mu_t, t \geq 0)$. First, we describe dynamically the limit set of μ_t .

Theorem 6.6. *The limit set of $\{\mu_t, t \geq 0\}$ is $\mathbb{P}_{x,r,\mu}$ -almost surely an attractor-free set of Φ .*

Proof. It results from Theorem 3.6 and [3]. □

Corollary 6.7. $\mathbb{P}_{x,r,\mu} \left(\overline{\lim}_{t \rightarrow +\infty} |X_t| = +\infty \right) = 1.$

Proof. Let A be a open subset of \mathbb{R}^d such that $\gamma(A) > 0$. Since the measure γ is diffusive, we have that for all $\nu \in \widehat{\text{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$, there exist $m, M > 0$ (depending on β only) such that $m\gamma \leq \nu \leq M\gamma$. Now, if we consider a sequence $(\nu_{t_n}, n \geq 0)$ in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d; V))$, the limits of its convergent subsequences will belong to $\widehat{\text{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$, because $\omega(\mu_t, t \geq 0)$ is a.s. an attractor-free set of Φ . Thus, there exists a subsequence $(\nu_{t_{n_k}})$ such that $\nu_{t_{n_k}}$ converges almost surely to ν for the weak topology. For any smooth function φ compactly supported, we have that

$$\nu_{t_n}(\varphi) \xrightarrow{w} \nu(\varphi).$$

If we consider φ such that it equals 1 on A and 0 out of a set B containing A , we find that $\nu(\varphi) \geq \nu(A) > 0$. Thus

$$\nu(B) \geq \limsup \nu_t(\varphi) \geq \liminf \nu_t(\varphi) \geq \nu(A) \geq m\gamma(A).$$

So, it implies that $\int_0^{t_n} \delta_{X_s}(A) ds$ is asymptotically equivalent to $t_n m\gamma(A)$, which in turn gives $\int_0^\infty \delta_{X_s}(A) ds = \infty$ a.s. Then, for all constant $K > 0$, $\int_0^\infty \delta_{X_s}(\mathbb{R}^d \setminus \overline{B}_K) ds = \infty$ a.s., where \overline{B}_K is the closed ball of radius K . Finally

$$\mathbb{P}_{x,r,\mu} \left(\bigcap_K \left\{ \int_0^\infty ds \mathbb{1}_{\{|X_s| \geq K\}} = \infty \right\} \right) = 1.$$

□

Second, we look at the (nonempty) set $\widehat{\text{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$.

Theorem 6.8. *$\widehat{\text{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$ is a.s. a global attracting set for Φ .*

Proof. We begin to notice that $\widehat{\text{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$ is a.s. compact for the weak topology and $\widehat{\text{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$ is positively invariant by definition. For all $\mu \in \omega(\mu_t, t \geq 0)$, we assert that $d(\Phi_t(\mu), \widehat{\text{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0))$ converges to 0 uniformly in μ . Indeed, recall that $\omega(\mu_t, t \geq 0)$ is an attractor free set for Φ , so for all $s \geq 0$, $\Phi_s(\mu) \in \omega(\mu_t, t \geq 0)$. As we also already know that $\lim d(\Phi_t(\mu), \widehat{\text{Im}(\Pi)}) = 0$, uniformly in μ , the assertion is proved. \square

Lemma 6.9. $\omega(\mu_t, t \geq 0)$ is a.s. a subset of $\widehat{\text{Im}(\Pi)}$.

Proof. As $\mu_{h(t)}$ is an asymptotic pseudotrajectory for the semiflow, which implies that $\omega(\mu_t, t \geq 0)$ is attractor free, we have by Theorem 6.8 that $\omega(\mu_t, t \geq 0)$ is the only attractor of Φ restricted to this set. Therefore, $\widehat{\text{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0) = \omega(\mu_t, t \geq 0)$. Consequently, $\omega(\mu_t, t \geq 0) \subset \widehat{\text{Im}(\Pi)}$. \square

When W is symmetric, we can give a better description of $\omega(\mu_t, t \geq 0)$. Let begin with the following:

Theorem 6.10. (Tromba [22]) Let \mathcal{B} be a C^∞ Banach manifold, F a C^∞ vector field on \mathcal{B} and $\mathcal{E} : \mathcal{B} \rightarrow \mathbb{R}$ a C^∞ function. Assume that:

- (1) $D\mathcal{E}(\mu) = 0$ if and only if $F(\mu) = 0$;
- (2) $F^{-1}(0)$ is compact;
- (3) for each $\mu \in F^{-1}(0)$, $D\mathcal{E}(\mu)$ is a Fredholm operator.

Then $\mathcal{E}(F^{-1}(0))$ has an empty interior.

Proposition 6.11. ([2], proposition 6.4) Let Λ be a compact invariant set for a semiflow Φ on a metric space E . Assume that there exists a continuous function $\mathcal{V} : E \rightarrow \mathbb{R}$ such that:

- (1) $\mathcal{V}(\Phi_t(x)) < \mathcal{V}(x)$ for $x \in E \setminus \Lambda$ and $t > 0$;
- (2) $\mathcal{V}(\Phi_t(x)) = \mathcal{V}(x)$ for $x \in \Lambda$ and $t > 0$.

If \mathcal{V} has an empty interior, then every attractor-free set A for Φ is contained in Λ . Furthermore, \mathcal{V} restricted to A is constant.

Theorem 3.8 . Suppose that W is symmetric. Then the limit set $\omega(\mu_t, t \geq 0)$ is $\mathbb{P}_{x,r,\mu}$ -a.s. a compact connected subset of the fixed points of Π .

Proof. We work only with absolutely continuous probability measures. We want to use Proposition 6.11 with the Lyapunov function \mathcal{E} (the free energy composed with Π), which satisfies the required condition. Lemma 4.7 asserts that the fixed points of Π form a nonempty compact subset of $\mathcal{P}(\mathbb{R}^d; V)$. Let $F(\mu) := \Pi(\mu) - \mu$. We already know that $F^{-1}(0)$ is compact for the weak topology. Therefore, we only need to show that $\mathcal{E}(F^{-1}(0))$ has an empty interior. Let $\mu \in F^{-1}(0)$ and prove that $DF(\mu)$ is a Fredholm operator. Let $\nu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$. Thanks to Lemma 2.1, there exists a constant $C(\beta)$ such that $\|DF(\mu) \cdot \nu\|_V \leq C(\beta)\|\nu\|_V$. So, the set $\{DF(\mu) \cdot \nu; \|\nu\|_V \leq 1\}$ is bounded. For $x, y \in \mathbb{R}^d$, we get

$$\begin{aligned} |DF(\mu) \cdot \nu(x) - DF(\mu) \cdot \nu(y)| &\leq 2|W * \nu(x)\Pi(\mu)(x) - W * \nu(y)\Pi(\mu)(y)| \\ &\quad + 2 \int W * \nu d\Pi(\nu)|(\Pi(\mu)(x) - \Pi(\mu)(y))| \\ &\leq M(|x - y|\|\mu\|_V + |\mu(x) - \mu(y)|) \\ &\quad + |V(x) - V(y)| + \|W(y, \cdot) - W(x, \cdot)\|_V \|\mu\|_V \end{aligned}$$

So, the map $DF(\mu) \cdot \nu$ ($\|\nu\|_V \leq 1$) is equicontinuous and by Ascoli's theorem, we conclude that the preceding set is relatively compact in $\mathcal{C}^0(\mathbb{R}^d; V)$ and thus the operator $DF(\mu)$ is compact. Moreover, this operator is self-adjoint. It follows from the spectral theory of compact self-adjoint operators that DF has at most countably many real eigenvalues; the set of nonzero eigenvalues is either finite or can be ordered as $|\lambda_1| > |\lambda_2| > \dots > 0$ with $\lim_{n \rightarrow \infty} \lambda_n = 0$. Therefore, we apply the result of Tromba and $\mathcal{E}(F^{-1}(0))$ has an empty interior. We conclude thanks to Proposition 6.11. \square

7. ILLUSTRATION IN DIMENSION $d = 2$

When W is not symmetric, it can happen that there exists no Lyapunov function and that the limit set $\omega(\mu_t, t \geq 0)$ is a non trivial orbit. Suppose for instance that (for $d = 2$) $W(x, y) = (x, Ry)$ where R is a rotation matrix, V is a polynomial $V(x) = V(|x|) := a|x|^4 + b|x|^2 + 1$. Note, that the probability measure $\gamma(dx) = e^{-2V(x)}dx/Z$ is invariant by rotation. Then, one expects, depending on R and V , that either the unique invariant set for the semiflow is $\{\gamma\}$ and so μ_t converges a.s. to γ ; or μ_t converges a.s. to a random measure, related to the critical points of the free energy; or $\omega(\mu_t, t \geq 0)$ is a periodic orbit related to γ . Remark that, equivalently considering $W(x, y) + \frac{1}{2}(b|x|^2 + |y|^2/b)$ or W , we satisfy the set of conditions **(H)**. We denote $p := (1, 0)^T$.

Lemma 7.1. ([4], lemma 4.6) *For all continuous $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, for all $y \in \mathbb{S}^1$ we have*

$$\int_{\mathbb{R}^2} [\varphi((x, y)) - \varphi((x, p))] \gamma(dx) = \int_{\mathbb{R}^2} \varphi((x, y))(x - (x, y)y) \gamma(dx) = 0.$$

Proof. For all $y \in \mathbb{S}^1$, there exists $g \in O(2)$ such that $y = gp$. The first equality follows from a change of variable in the integral (because $V(x) = V(|x|)$). After, define $\phi(y) := \int_{\mathbb{R}^2} \varphi((x, y))(x - (x, y)y) \gamma(dx)$. We clearly have $(\phi(y), y) = 0$ and the rotation-invariance of γ implies for the antisymmetry matrix h , $\phi(p) = h\phi(p)$. So, $\phi(p) = 0$ and thus $\phi(y) = 0$. \square

For any probability measure $\mu \in \mathcal{P}(\mathbb{R}^2; V)$, define the mean of μ by $\bar{\mu} := \int_{\mathbb{R}^2} x \mu(dx)$. Let the probability measure

$$(7.1) \quad \bar{\Pi}(\bar{\mu})(dx) := \frac{e^{-2(x, R\bar{\mu})}}{Z(\bar{\mu})} \gamma(dx).$$

Here, $\bar{\Pi}(\bar{\mu}) = \Pi(\mu)$. If we let $\bar{\Pi}(\mu) := \int_{\mathbb{R}^2} x \bar{\Pi}(\mu)(dx)$, then $\bar{\Phi}_t(\mu)$ is readily the semiflow corresponding to

$$(7.2) \quad \bar{\Phi}_t(\mu) = e^{-t} \bar{\mu} + e^{-t} \int_0^t e^s \bar{\Pi}(\bar{\Phi}_s(\mu)) ds, \quad \bar{\Phi}_0(\mu) = \bar{\mu}.$$

Lemma 7.2. *Let $m = \rho v$ with $\rho \geq 0$ and $v \in \mathbb{S}^1$. Then we get*

$$\int_{\mathbb{R}^2} x \bar{\Pi}(m)(dx) = -\frac{1}{2} \frac{d}{d\rho} \log \left(\int_{\mathbb{R}^2} e^{-2\rho(x, v)} \gamma(dx) \right) Rv.$$

Proof. One just has to differentiate the function $\alpha \mapsto \log \left(\int_{\mathbb{R}^2} e^{-2\alpha(x, v)} \gamma(dx) \right)$ and use the second equality of Lemma 7.1. \square

Let $m = \rho v$ be the solution to the ODE $\dot{m} = \bar{\Pi}(m) - m$, with $\rho = |m|$ and $v \in \mathbb{S}^1$. Then we have by Lemma 7.2 that $\dot{v} = 0$. Moreover, if we let $\alpha = 2\rho$, then α satisfies the one-dimensional ODE

$$(7.3) \quad \dot{\alpha} = J(\alpha) = -\alpha + 2\partial_\alpha \log \left(\int_{\mathbb{R}^2} e^{-\alpha(x, Rp)} \gamma(dx) \right).$$

Let us define some useful functions expressed in polar coordinates:

$$\begin{aligned} H(\alpha) &:= \int_0^\infty d\rho \gamma(\rho) \int_0^{2\pi} dv e^{-\alpha \rho \cos v} \\ \tilde{H}(\alpha) &:= \int_0^\infty d\rho \gamma(\rho) \rho^2 \int_0^{2\pi} dv \sin^2 v e^{-\alpha \rho \cos v}. \end{aligned}$$

7.1. The case $R = -Id$. Here, W is a symmetric function. Expressing the problem in polar coordinates, we get $J(\alpha) = -\alpha \left(1 - 2\frac{\tilde{H}(\alpha)}{H(\alpha)}\right)$.

Proposition 7.3. *If $\int_0^\infty \rho^2 \gamma(\rho) d\rho \leq 1$, then 0 is the unique equilibrium of (7.3) and 0 is stable. The basin of attraction of 0 is \mathbb{R}_+ .*

If $\int_0^\infty \rho^2 \gamma(\rho) d\rho > 1$, then 0 is linearly unstable and there is another stable equilibrium α_1 , whose basin of attraction is \mathbb{R}_+^ .*

Proof. We remark that J is \mathcal{C}^∞ . A computation yields to

$$J^{(3)}(\alpha) = 2\frac{H^{(4)}(\alpha)}{H(\alpha)} - 8\frac{H^{(3)}(\alpha)}{H(\alpha)}\frac{H'(\alpha)}{H(\alpha)} + 24\frac{H''(\alpha)}{H(\alpha)}\left(\frac{H'(\alpha)}{H(\alpha)}\right)^2 - 12\left(\frac{H'(\alpha)}{H(\alpha)}\right)^4.$$

We wonder for the sign of $J^{(3)}$. This function corresponds to (twice) the kurtosis of the projection on the axis x of a random variable X (expressed in polar coordinates) such that X has the law γ . As the graph of the symmetric part of the density function cuts exactly twice the graph of the corresponding Gaussian variable (with the same mean and variance), the kurtosis of X is negative, or more exactly $J^{(3)}(\alpha) < 0$ for $\alpha > 0$ and $J^{(3)}(0) = 0$. So, for all $\alpha \geq 0$, we have $J''(\alpha) \leq J''(0) = 0$. Similarly, we find

$$J'(\alpha) \leq J'(0) = -1 + \int_0^\infty d\rho \gamma(\rho) \rho^2.$$

Therefore, if $J'(0) \leq 0$, then J is a decreasing function and as $J(0) = 0$, the first result follows. Else $J'(0) > 0$. But J' is a non-increasing function and $\lim_{\alpha \rightarrow \infty} J'(\alpha) = -1$. So, because of the continuity of J' , there exists $\alpha_0 > 0$ such that $J'(\alpha_0) = 0$. Moreover, we have $\lim_{\alpha \rightarrow \infty} J(\alpha) = -\infty$. Finally, there exists a positive solution to $J(\alpha) = 0$ if and only if $\int_0^\infty d\rho \gamma(\rho) \rho^2 > 1$. In that case, the point 0 is unstable and there exists an other equilibrium, which is stable. \square

Remark 7.4. *The function $t \mapsto \int_0^{2\pi} e^{-t \cos v} dv$ is the Bessel function $I_0(t)$.*

The next result shows that we can reduce the problem in studying the dynamical system satisfied by $\bar{\mu}$ and then deduce results on μ .

Lemma 7.5. ([4] proposition 3.9, corollary 3.10) 1) *Let $L \subset \mathcal{P}_\beta(\mathbb{R}^d; V)$ be an attractor-free set for Φ and $A \subset \mathcal{P}_\beta(\mathbb{R}^d; V)$ an attractor for Φ . If $L \cap B(A) \neq \emptyset^1$, then $L \subset A$.*

2) *Let (E, d) be a metric space, $\bar{\Phi} : E \times \mathbb{R} \rightarrow E$ a semiflow on E and $G : \mathcal{P}_\beta(\mathbb{R}^d; V) \rightarrow E$ a continuous function. Assume that $G \circ \Phi_t = \bar{\Phi}_t \circ G$. Then almost surely $G(\omega(\mu_t, t \geq 0))$ is an attractor-free set of $\bar{\Phi}$.*

We can now state and prove the following

Theorem 7.6. *Consider the self-interacting diffusion on \mathbb{R}^2 associated with $W(x, y) = -(x, y)$. Then we have two different cases:*

- (1) *If $\int_0^\infty d\rho \gamma(\rho) \rho^2 \leq 1$, then a.s. $\mu_t \xrightarrow{(w)} \gamma$;*
- (2) *If $\int_0^\infty d\rho \gamma(\rho) \rho^2 > 1$, then there exists a random variable $v \in \mathbb{S}^1$ such that a.s. $\mu_t \xrightarrow{(w)} \mu_\infty^v$ with*

$$\mu_\infty^v(dx) = \frac{e^{\alpha_1(x, v)}}{Z_1} \gamma(dx),$$

where Z_1 is the normalization constant and α_1 is the unique positive solution to the equation $J(\alpha) = -\alpha + 2\frac{H'(\alpha)}{H(\alpha)} = 0$.

¹ $B(A)$ is the basin of attraction of A

Proof. Let $G : \mathcal{P}_\beta(\mathbb{R}^2; V) \rightarrow \mathbb{R}^2$ be the mapping defined by $G(\mu) = \bar{\mu}$. By Lemma 7.5, the limit set of $\bar{\mu}_t$ is a.s. an attractor-free set of $\bar{\Phi}$. When $\int_0^\infty d\rho\gamma(\rho)\rho^2 \leq 1$, then 0 is a global attractor for the dynamical system generated by $\bar{\Phi}$. Therefore, each attractor-free set of $\bar{\Phi}$ reduces to 0. So, a.s. $\bar{\mu}_t \xrightarrow{(w)} 0$ and $\omega(\mu_t, t \geq 0) \subset G^{-1}(0)$. The definitions of $\bar{\Pi}(\bar{\mu})$ and J imply that $G^{-1}(0)$ is invariant under the action of Φ and, as $\Pi(\Phi_t|_{G^{-1}(0)}(\mu)) = \gamma$, we have

$$\Phi|_{G^{-1}(0)}(\mu) = e^{-t}(\mu - \gamma) + \gamma.$$

Therefore, γ is a global attractor for $\Phi|_{G^{-1}(0)}$. Lemma 7.5 then implies that each attractor-free set reduces to γ . By Theorem 6.6, we conclude that $\omega(\mu_t, t \geq 0) = \gamma$.

Suppose now that 0 is unstable for $\bar{\Pi} - Id$. For all $f \in \mathcal{C}^\infty(\mathbb{R}^2; V)$, it holds

$$\frac{d}{dt}\mu_{h(t)}f = -\mu_{h(t)}f + \Pi(\mu_{h(t)})f + \frac{d}{ds}\varepsilon_{t,t+s}|_{s=0}f.$$

If we consider the projection map $P_i(x) = x_i$, then $\partial_t \bar{\mu}_{h(t)} = \bar{\Pi}(\bar{\mu}_{h(t)}) - \bar{\mu}_{h(t)} + \eta_t$ where η_t is the random vector $\eta_t = \frac{d}{ds}\varepsilon_{t,t+s}|_{s=0}(P_1, P_2)^T$. As 0 is an unstable linear equilibrium for $\bar{\Pi} - Id$, we apply the result of Tarrès ([21], part 3) to prove that $\mathbb{P}(\lim_{t \rightarrow \infty} \bar{\mu}_{h(t)} = 0) = 0$. Thanks to Theorem 3.6, we obtain that $\lim_{t \rightarrow \infty} \lim_{0 \leq s \leq T} |\bar{\mu}_{h(t+s)} - \bar{\Phi}_s(\bar{\mu}_{h(t)})| = 0$. We remind that

$$(7.4) \quad \dot{\rho} = -\rho - \frac{H'(\alpha)}{H(\alpha)}$$

and we denote by α_1 the unique positive solution to $-\alpha + 2\frac{H'(\alpha)}{H(\alpha)} = 0$. We introduce the invariant set (for $\bar{\Phi}$) $A := \{m = \rho v; \rho = \alpha_1, v \in \mathbb{S}^1\}$. As the limit set of $\bar{\mu}_{h(t)}$ is an attractor-free set by Lemma 7.5, the ODE (7.4) implies that $\omega(\bar{\mu}_{h(t)})$ either reduces to $\{0\}$, or is included in A . But as $\mathbb{P}\left(\lim_{t \rightarrow \infty} \bar{\mu}_{h(t)} = 0\right) = 0$, the limit set of $\bar{\mu}_{h(t)}$ is a.s. a subset of A . Moreover, as $\dot{v} = 0$, we have $\bar{\Phi}_t|_A = Id|_A$. So, $\bar{\mu}_{h(t)}$ is a Cauchy sequence in A and then there exists $v \in \mathbb{S}^1$ such that

$$\lim_{t \rightarrow \infty} |\bar{\mu}_{h(t)} - \alpha_1 v| = 0.$$

To conclude, we have on one side that the limit set of (μ_t) is an attractor-free set for $\Phi|_{G^{-1}(\alpha_1 v)}$ and on the other side, that the semiflow $\Phi|_{G^{-1}(\alpha_1 v)}$ admits μ_∞^v as a global attractor. This leads to $\omega(\mu_t, t \geq 0) = \mu_\infty^v$. \square

7.2. The case “ R is a rotation”. We assume that $R = R(\theta)$ is defined by $R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, with $0 \leq \theta < 2\pi$. We emphasize that (unless $\theta = 0, \pi$) W is not a symmetric function.

Theorem 7.7. *Consider the self-interacting diffusion on \mathbb{R}^2 associated with $W(x, y) = (x, Ry)$. Then one of the following holds:*

- (1) *If V is such that $\int_0^\infty d\rho\gamma(\rho)\rho^2 \cos(\theta) > -1$, then a.s. $\mu_t \xrightarrow{(w)} \gamma$;*
- (2) *If V is such that $\int_0^\infty d\rho\gamma(\rho)\rho^2 \cos(\theta) \leq -1$, then we get two cases:*
 - a) *if $\theta = \pi$ then there exists a random variable $v \in \mathbb{S}^1$ such that a.s. $\mu_t \xrightarrow{(w)} \mu_\infty^v$ with $\mu_\infty^v(dx) = \frac{e^{\alpha_1(x, v)}}{Z_1} \gamma(dx)$, where Z_1 is the normalization constant and α_1 is the unique positive solution to $-\alpha + 2\frac{H'(\alpha)}{H(\alpha)} = 0$,*
 - b) *if $\theta \neq \pi$, then $\omega(\mu_t, t \geq 0) = \{\nu(\delta), 0 \leq \delta < 2\pi\}$ a.s., where $\nu(\delta) = \frac{1}{e^{T_\theta}-1} \int_0^{T_\theta} e^s \mu_\infty^{v, \theta} ds$, with $T_\theta = 2\pi(\tan \theta)^{-1}$ and $\mu_\infty^{v, \theta}$ is the unique positive solution to $-\alpha + 2 \cos \theta \frac{H'(\alpha)}{H(\alpha)} = 0$.*

Proof. Let $v = gp$ with $g \in O(2)$ and $m = \alpha v/2$. We remind the equations

$$\dot{\alpha} = -\alpha - 2 \frac{H'(\alpha)}{H(\alpha)} (Rv, v); \quad \dot{v} = -\frac{2}{\alpha} \frac{H'(\alpha)}{H(\alpha)} ((Rv, v)v - Rv).$$

But, by definition of R and $v = (\cos \sigma, \sin \sigma)^T$, a simple computation yields to

$$(Rv, v)v - Rv = \begin{pmatrix} -\sin \theta \sin \sigma \\ \sin \theta \cos \sigma \end{pmatrix}.$$

We finally get after some easy calculations

$$(7.5) \quad \begin{cases} \dot{\alpha} = -\alpha - \frac{2H'(\alpha)}{H(\alpha)} \cos \theta; \\ \dot{\sigma} = \frac{2H'(\alpha)}{\alpha H(\alpha)} \sin \theta. \end{cases}$$

We recall that $\frac{H'(\alpha)}{H(\alpha)} > 0$ for $\alpha > 0$. By Proposition 7.3, we have a bifurcation at $\cos \theta \int_0^\infty \gamma(d\rho) \rho^2 = 1$. More precisely, if $\cos \theta \int_0^\infty \gamma(d\rho) \rho^2 \geq 1$, then the set $\{(\sigma, \alpha); \alpha = 0\}$ is a global attracting set for Equation (7.5) and so a.s. $\mu_t \xrightarrow{(w)} \gamma$. If $\cos \theta \int_0^\infty \gamma(d\rho) \rho^2 < 1$, then $\{(\sigma, \alpha); \alpha = \alpha_1(\cos \theta)\}$ is a global attracting set. On this set, the dynamics is given by

$$\dot{\sigma} = \frac{2H'(\alpha_1(\cos \theta))}{\alpha_1(\cos \theta)H(\alpha_1(\cos \theta))} \sin \theta = \tan \theta.$$

By Theorem 7.6, we show that there exists a random variable σ_0 such that a.s.

$$(7.6) \quad \lim_{t \rightarrow \infty} \left| \bar{\mu}_{h(t)} - \frac{\alpha_1(\cos \theta)}{2} v(t \tan \theta + \sigma_0) \right| = 0.$$

At that point, we know the dynamics on the set $\tilde{A} := \{(\sigma, \alpha); \alpha = \alpha_1(\cos \theta)\}$. Unfortunately, we need more to conclude: we have to study the coupled system defined on $\mathcal{M}(\mathbb{R}^2; V) \times \mathbb{R}^2$ by

$$(7.7) \quad \begin{cases} \dot{m} = -m + \bar{\Pi}(m); \\ \dot{\nu} = -\nu + \bar{\Pi}(m). \end{cases}$$

By Lemma 7.5, $\omega(\mu_t, t \geq 0) \times \tilde{A}$ is an attractor-free set for the preceding semiflow restricted to $\mathcal{P}(\mathbb{R}^2; V) \times \mathbb{R}^2$. The dynamics on $\omega(\mu_t, t \geq 0) \times \tilde{A}$ is given by

$$(7.8) \quad \begin{cases} \dot{\sigma} = \tan \theta; \\ \dot{\nu} = -\nu + f(\sigma) = -\nu + \mu_\infty^{v, \cos \theta}. \end{cases}$$

As the set $\omega(\mu_t, t \geq 0) \times \tilde{A}$ is compact (for the weak topology) and invariant in $\mathcal{P}(\mathbb{R}^2; V) \times \mathbb{R}^2$, we conclude by following the lines of [4] (theorem 4.11). \square

REFERENCES

- [1] BAKRY D. [1994], L'hypercontractivité et son utilisation en théorie des semigroupes, *Lectures on Prob. Th. and Stat., Ecole de Prob. de St-Flour*, Springer, 1-114.
- [2] BENAÏM M. [1999], Dynamics of stochastic approximation algorithms, *Sém. Prob. XXXIII, Lecture Notes in Math.* **1709**, 1-68, Springer.
- [3] BENAÏM M. & HIRSCH M.W. [1996], Asymptotic pseudotrajectories and chain recurrent flows, *J. Dynam. Diff. Equ.* **8**, 141-176.
- [4] BENAÏM M., LEDOUX M. & RAIMOND O. [2002], Self-interacting diffusions, *Prob. Theory Relat. Fields* **122**, 1-41.
- [5] BENAÏM M. & RAIMOND O. [2005], Self-interacting diffusions III: symmetric interactions, *Ann. Prob.* **33**(5), 1716-1759.
- [6] CRANSTON M. & LE JAN Y. [1995], Self-attracting diffusions: two cases studies, *Math. Ann.* **303**, 87-93.
- [7] CRANSTON M. & MOUNTFORD T.S. [1996], The strong law of large numbers for a Brownian polymer, *Ann. Prob.* **2**(3), 1300-1323.
- [8] DAVIES E.B. & SIMON B. [1984], Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians, *J. Func. An.* **59**, 335-395.

- [9] DURRETT R.T. & ROGERS L.C.G. [1992], Asymptotic behavior of Brownian polymers, *Prob. Th. Rel. Fields* **92**(3), 337-349.
- [10] HERRMANN S. & ROYNETTE B. [2003], Boundedness and convergence of some self-attracting diffusions, *Math. Ann.* **325**(1), 81-96.
- [11] KONTOYIANNIS I. & MEYN S.P. [2003], Spectral Theory and Limit Theory for Geometrically Ergodic Markov Processes, *Ann. App. Prob.* **13**, 304-362.
- [12] KUNITA H. [1990], *Stochastic flows and stochastic differential equations*, Cambridge studies in advanced mathematics.
- [13] LEDOUX M. [2000], The geometry of Markov diffusion generators, *Ann. Fac. Sci. Toulouse* **IX**, 305-366.
- [14] MCCANN R. [1997], A convexity principle for interacting gases, *Adv. Math.* **128**, 153-179.
- [15] MEYN S.P. & TWEEDIE R.L. [1993], *Markov Chains and Stochastic Stability*, Springer-Verlag.
- [16] MOUNTFORD T.S. & TARRÈS P. [2007], An asymptotic result for Brownian polymers, *Ann. Inst. H. Poincaré, Prob. Stat.*, to appear.
- [17] PEMANTLE R. [2007], A survey of random processes with reinforcement, *Prob. Surveys* **4**, 1-79.
- [18] RAIMOND O. [1997], Self-attracting diffusions: case of constant interaction, *Prob. Theory Relat. Fields* **107**, 177-196.
- [19] RÖCKNER M. & WANG F.Y. [2003], Supercontractivity and Ultracontractivity for (Nonsymmetric) Diffusions Semigroups on Manifolds, *Forum Math.* **15**, 893-921.
- [20] ROTHUS O. [1981], Diffusions on compact Riemannian manifolds and logarithmic Sobolev inequalities, *J. Func. An.* **42**, 102-109.
- [21] TARRÈS P. [2000], Pièges répulsifs, *C. R. Acad. Sci. Paris Sér. I Math.* **330**, 125-130.
- [22] TROMBA A.J. [1977], The Morse-Sard-Brown theorem for functionals and the problem of Plateau, *Amer. J. Math.* **99**, 1251-1256.
- [23] VILLANI C. [2003], *Topics in Optimal Transportation*, Graduate Studies in Mathematics, Vol. 58, AMS.

Aline KURTZMANN, Oxford University, Mathematical Institute, 24-29 St Giles', Oxford, OX1 3LB, United Kingdom. kurtzmann@maths.ox.ac.uk